MATH 251, Section ______
Thursday, Nov. 11, 2010

Quiz 12 (Sections 14.4, 14.5).
Dr. M. Vorobets

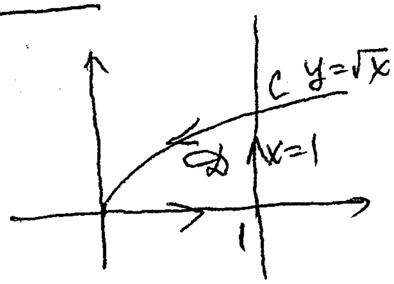
Due Tuesday, Nov. 16 at the beginning of class.

NAME (print): key

No credit for unsupported answers will be given. Clearly indicate your final answer.

1. *[3 pts.]* Use Green's Theorem to evaluate $\oint_C (1+\tan x)dx + (x^2+e^y)dy$ where C is the positively oriented boundary of the region enclosed by the curves $y=\sqrt{x}$, $x=1$, and $y=0$.

Green's Thm. $\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$.



$Q(x,y) = x^2 + e^y \qquad P(x,y) = 1 + \tan x$

$Q_x = 2x \qquad P_y = 0.$

$$\oint_C (1+\tan x)dx + (x^2+e^y)dy = \iint_D 2x\,dA$$

$0 \le x \le 1$
$0 \le y \le \sqrt{x}$

$$= 2\int_0^1\int_0^{\sqrt{x}} x\,dy\,dx = 2\int_0^1 x\,y\Big|_{y=0}^{y=\sqrt{x}} dx$$

$$= 2\int_0^1 x\sqrt{x}\,dx = 2\,\frac{x^{5/2}}{5/2}\Big|_0^1 = \boxed{\frac{4}{5}}$$

2. *[2 pts.]* Find the divergence of the vector field $\vec{F} = e^{xyz}\vec{\imath} + \sin(x-y)\vec{\jmath} - \frac{xy}{z}\vec{k}$.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(e^{xyz}\right) + \frac{\partial}{\partial y}\left(\sin(x-y)\right) + \frac{\partial}{\partial z}\left(-\frac{xy}{z}\right)$$

$$= \boxed{yze^{xyz} - \cos(x-y) + \frac{xy}{z^2}}$$

[more problems on back]

3. *[5 pts.]* Show that $\vec{F}(x,y,z) = yz(2x+y)\vec{\imath} + xz(x+2y)\vec{\jmath} + xy(x+y)\vec{k}$ is conservative and use this fact to evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the curve C given by $\vec{r}(t) = (1+t)\vec{\imath} + (1+2t^2)\vec{\jmath} + (1+3t^3)\vec{k}$, $0 \le t \le 1$.

$\vec{r}(0) = \langle 1, 1, 1 \rangle$

$\vec{r}(1) = \langle 2, 3, 4 \rangle$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz(2x+y) & xz(x+2y) & xy(x+y) \end{vmatrix} = \vec{\imath}\left(\frac{\partial}{\partial y}(xy(x+y)) - \frac{\partial}{\partial z}(xz(x+2y))\right)$$

$$-\vec{\jmath}\left(\frac{\partial}{\partial x}(xy(x+y)) - \frac{\partial}{\partial z}(yz(2x+y))\right) + \vec{k}\left(\frac{\partial}{\partial x}(xz(x+2y)) - \frac{\partial}{\partial y}(yz(2x+y))\right)$$

$$= \vec{\imath}\left(x^2+2xy-(x^2+2xy)\right) - \vec{\jmath}\left(2xy+y^2-(2xy+y^2)\right) + \vec{k}\left(2xz+2yz-(2xz+2yz)\right) = \vec{0}$$

$\vec{F}$ is conservative.

f such that $\nabla f = \vec{F}$:

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xyz + y^2z, x^2z+2xyz, x^2y+xy^2 \rangle$

$$\begin{cases} f_x = 2xyz + y^2z \\ f_y = x^2z + 2xyz \\ f_z = x^2y + xy^2 \end{cases}$$

$f(x,y,z) = \int f_x\,dx = \int (2xyz + y^2z)\,dx$

$= x^2yz + xy^2z + g(y,z)$

g is unknown.

Plug f into the 2nd and 3rd equations.

$f_y = x^2z + 2xyz + g_y = x^2z + 2xyz$

$g_y = 0$

$f_z = x^2y + xy^2 + g_z = x^2y + xy^2$

$g_z = 0$

$g(y,z) = K$ – constant.

$f(x,y,z) = x^2yz + xy^2z + K$.

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(2,3,4) - f(1,1,1)$

$= (4)(3)(4) + (2)(9)(4) + K - 1 - 1 - K = \boxed{118}$