Solutions for the problems 43–50.

43. Find an equation of the tangent plane to the surface given by parametric equations $x = u^2$, $y = u - v^2$, $z = v^2$, at the point (1,0,1).

$$\vec{r}(u,v) = \langle u^2, u-v^2, z=v^2 \rangle$$

$$\vec{r}_u = \langle 2u, 1, 0 \rangle$$

$$\vec{r}_v = \langle 0, -2v, 2v \rangle$$

point $(1,0,1) = \vec{r}(1,1)$.

$$\vec{r}_u(1,1) = \langle 2,1,0 \rangle$$

$$\vec{r}_v(1,1) = \langle 0,-2,2 \rangle.$$

$$\vec{r}_u(1,1) \times \vec{r}_v(1,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix}$$

$$= 2\vec{i} - 4\vec{j} - 4\vec{k}$$

Thus, the equation of the tangent plane is

$$\boxed{2(x-1) - 4(y-0) - 4(z-1) = 0}$$

$$2x - 2 - 4y - 4z + 4 = 0.$$

$$2x - 4y - 4z + 2 = 0$$

or $\boxed{x - 2y - 2z + 1 = 1}$

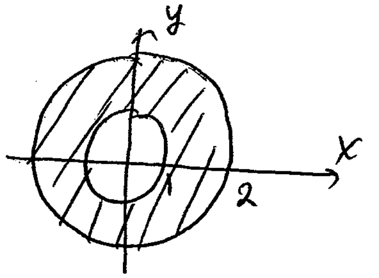
44. Find the area of the hyperbolic paraboloid $z = x^2 - y^2$ that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

$$A = \iint_{D} \sqrt{1+\left[\frac{\partial z}{\partial x}\right]^2+\left[\frac{\partial z}{\partial y}\right]^2}\, dA = \iint_{D} \sqrt{1+4x^2+4y^2}\, dA$$

where $z = x^2 - y^2$

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$$

D:



D is bounded by

$x^2+y^2=1$

$x^2+y^2=4$.

polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2\cos^2\theta+4r^2\sin^2\theta}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_1^2 \sqrt{1+4r^2}\; r\,dr$$

$$= 2\pi \int_1^2 \sqrt{1+4r^2}\; r\,dr$$

substitution:

$1+4r^2 = u$

$du = 8r\,dr$

$r=1 \to u=5$

$r=2 \to u=17$

$$= 2\pi\cdot\frac{1}{8}\int_5^{17}\sqrt{u}\,du$$

$$= \frac{\pi}{4}\cdot\left.\frac{u^{3/2}}{3/2}\right|_5^{17}$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}\left[17^{3/2}-5^{3/2}\right]$$

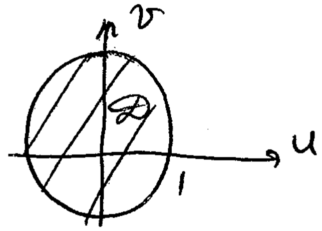
$$= \boxed{\frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)}$$

45. Find the area of the surface with parametric equations $x = uv$, $y = u+v$, $z = u-v$, $u^2+v^2 \le 1$.

$$\vec{r}(u,v) = \langle uv,\ u+v,\ u-v \rangle, \qquad u^2+v^2 \le 1.$$

$$A = \iint_{D} |\vec{r}_u \times \vec{r}_v|\, dA$$

$D:\ u^2+v^2 \le 1$



$$\vec{r}_u = \langle v, 1, 1 \rangle$$
$$\vec{r}_v = \langle u, 1, -1 \rangle$$
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix}$$
$$= \vec{i}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} v & 1 \\ u & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} v & 1 \\ u & 1 \end{vmatrix}$$
$$= -2\vec{i} + \vec{j}(u+v) + \vec{k}(v-u)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4+(u+v)^2+(v-u)^2} = \sqrt{4+u^2+2uv+v^2+v^2-2uv+v^2}$$
$$= \sqrt{4+2u^2+2v^2} = \sqrt{2}\sqrt{2+u^2+v^2}$$

$$A = \sqrt{2}\iint_{D} \sqrt{2+u^2+v^2}\, dA = \left|\begin{array}{l} u = r\cos\theta \\ u = r\sin\theta \\ dA = r\,dr\,d\theta \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}\right|$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \sqrt{2+r^2}\, r\, dr\, d\theta = \sqrt{2}\int_0^{2\pi} d\theta \cdot \int_0^1 \sqrt{2+r^2}\, r\, dr$$

$$= \sqrt{2}(2\pi)\int_0^1 \sqrt{2+r^2}\, r\, dr \quad \left|\begin{array}{l} 2+r^2 = t \\ dt = 2r\,dr \\ r=0 \to t=2 \\ r=1 \to t=3 \end{array}\right.$$

$$= 2\sqrt{2}\pi \frac{1}{2}\int_2^3 \sqrt{t}\, du = \sqrt{2}\pi \left.\frac{t^{3/2}}{3/2}\right|_2^3 = \boxed{\frac{2\sqrt{2}\pi}{3}\left(3^{3/2} - 2^{3/2}\right)}.$$

46. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \le z \le 4$ if its density function is $\rho(x,y,z) = 10 - z$.

$$m = \iint_S \rho(x,y,z)\,dS.$$

$S:\quad x = x$
$\quad\quad y = y$
$\quad\quad z = \sqrt{x^2+y^2}$

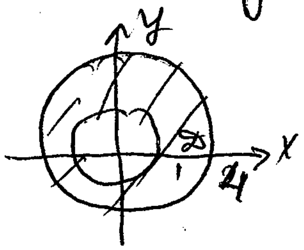
$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}};\quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1+\left[\frac{\partial z}{\partial x}\right]^2+\left[\frac{\partial z}{\partial y}\right]^2} = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}} = \sqrt{2}$$

$$m = \iint_D \left(10-\sqrt{x^2+y^2}\right)\sqrt{2}\,dA$$

Polar coordinates:
$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$
$1 \le r \le 4$
$0 \le \theta \le 2\pi$

$D:\quad x^2+y^2 = 1$
$\quad\quad x^2+y^2 = 4^2$



$$= \int_0^{2\pi}\int_1^4 (10-r)\sqrt{2}\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi} d\theta \cdot \int_1^4 (10r - r^2)\,dr$$

$$= \sqrt{2}\,\theta\Big|_0^{2\pi} \cdot \left(\frac{10r^2}{2} - \frac{r^3}{3}\right)\Big|_1^4$$

$$= 2\sqrt{2}\pi\left(5(16) - \frac{64}{3} - 5 + \frac{1}{3}\right)$$

$$= 2\sqrt{2}\pi\left(75 - \frac{63}{3}\right)$$

$$= 54(2)\sqrt{2}\pi$$

$$= \boxed{108\sqrt{2}\pi}$$

47. Evaluate $\iint_S yz\,dS$ if S is the part of the plane $z = y+3$ that lies inside the cylinder $x^2+y^2=1$.

$$\iint_S yz\,dS = \iint_D y(y+3)\sqrt{1+\left[\frac{\partial z}{\partial x}\right]^2+\left[\frac{\partial z}{\partial y}\right]^2}\,dA$$

$S: \begin{cases} x = x \\ y = y \\ z = y+3 \end{cases}$

$$= \iint_D y(y+3)\sqrt{1+1}\,dA$$

$$= \iint_D y(y+3)\sqrt{2}\,dA$$

Polar coordinates

$x = r\cos\theta$

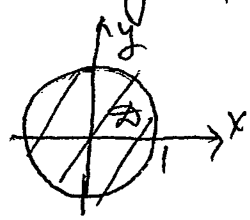
$y = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$D: x^2+y^2=1$



$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta + 3r\cos\theta\right) r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \left(r^3\cos^2\theta + 3r^2\cos\theta\right) dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{r^4}{4}\cos^2\theta + \frac{3r^3}{3}\cos\theta\right)\Bigg|_{r=0}^{r=1} d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{1}{4}\cos^2\theta + \cos\theta\right) d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{1}{8} + \frac{1}{8}\cos 2\theta + \cos\theta\right) d\theta$$

$$= \sqrt{2}\left(\frac{1}{8}\theta + \frac{1}{16}\sin 2\theta + \sin 2\theta\right)\Bigg|_0^{2\pi}$$

$$= \sqrt{2}\,\frac{1}{8}(2\pi) = \boxed{\frac{\pi\sqrt{2}}{4}}$$

48. Let T be the solid bounded by the paraboloids

$$z = x^2 + 2y^2, \text{ and } z = 12 - 2x^2 - y^2.$$

Let $\vec{F} = < x, y, z >$. Find the outward flux of $\vec{F}$ across the boundary surface of T.

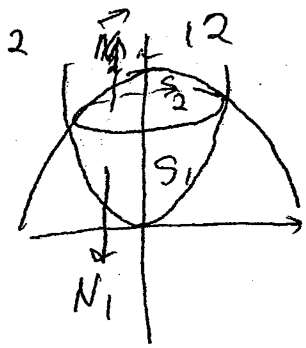
S: $z = x^2 + 2y^2$

$z = 12 - 2x^2 - y^2$

intersection:

$x^2 + 2y^2 = 12 - 2x^2 - y^2$

$\boxed{x^2 + y^2 = 4}$



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S}_1 + \iint_{S_2} \vec{F} \cdot d\vec{S}_2$$

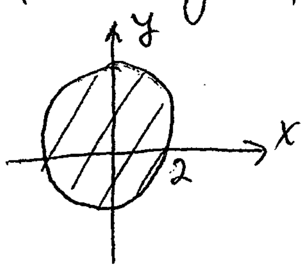
S_1: $\begin{cases} x = x \\ y = y \\ z = x^2 + 2y^2 \end{cases}$

$$\vec{N}_1 = -< -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1> = < 2x, 4y, -1>.$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \iint_D < x, y, x^2 + 2y^2 > \cdot < 2x, 4y, -1 > dA$$

$$= \iint_D (2x^2 + 4y^2 - x^2 - 2y^2)\, dA = \iint_D (x^2 + 2y^2)\, dA$$

D: $x^2 + y^2 = 4$



polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^2 (r^2\cos^2\theta + 2r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3(1 + \sin^2\theta)\, dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}(1 + \sin^2\theta)\Big|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} 4\left(1 + \frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} (4 + 2 - 2\cos 2\theta)\, d\theta = \int_0^{2\pi} (6 - 2\cos 2\theta)\, d\theta$$

$$= 6\left(\theta - 2 \cdot \frac{1}{2} \sin 2\theta\right)\Big|_0^{2\pi} = \boxed{6\pi}.$$

$$S_2: \begin{cases} x = x \\ y = y \\ z = 12 - 2x^2 - y^2 \end{cases}$$

$$\vec{N}_2 = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle = \langle 4x, 2y, 1 \rangle.$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \iint_{D} \langle x, y, 12 - 2x^2 - y^2 \rangle \cdot \langle 4x, 2y, 1 \rangle \, dA$$

$$= \iint_{D} (4x^2 + 2y^2 + 12 - 2x^2 - y^2) \, dA = \iint_{D} (2x^2 + y^2 + 12) \, dA$$

Polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_0^2 (2r^2\cos^2\theta + r^2\sin^2\theta + 12) \, \cancel{dA}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 (r^3 + r^3\cos^2\theta + 12r) \, dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^4}{4} + \frac{r^4}{4}\cos^2\theta + \frac{12r^2}{2} \right) \Bigg|_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} (4 + 4\cos^2\theta + 6(4)) \, d\theta = \int_0^{2\pi} (4 + 2(1 + \cos 2\theta) + 24) \, d\theta$$

$$= \int_0^{2\pi} (30 + 2\cos 2\theta) \, d\theta = \left(30\theta + \frac{2}{2}\sin 2\theta \right) \Bigg|_0^{2\pi} = \boxed{60\pi}$$

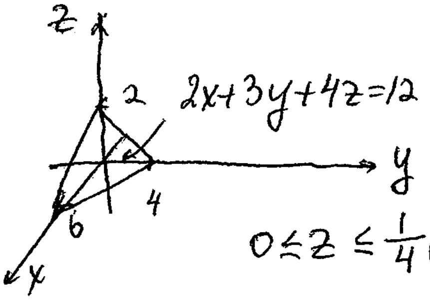
Thus,

$$\iint_S \vec{F} \cdot d\vec{S} = 6\pi + 60\pi = \boxed{66\pi}$$

49. Verify the Divergence Theorem for $\vec{F} = \langle x^2, xy, z \rangle$ and the region E bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$.

$$\iint_S \vec{F}\cdot d\vec{S} \stackrel{?}{=} \iiint_T \operatorname{div} \vec{F}\, dV$$

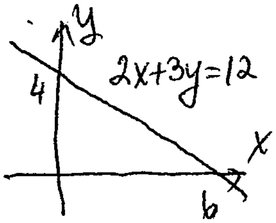
$$\operatorname{div} \vec{F} = 2x + x + 1 = 3x + 1$$



$$\iiint_T (3x+1)\,dV = \int_0^6 \int_0^{4-\frac{2}{3}x} \int_0^{3-\frac{3}{4}y-\frac{1}{2}x} (3x+1)\,dz\,dy\,dx$$

$0 \le z \le \frac{1}{4}(12 - 3y - 2x)$

$0 \le z \le 3 - \frac{3}{4}y - \frac{1}{2}x$



$0 \le y \le \frac{1}{3}(12 - 2x)$

$0 \le y \le 4 - \frac{2}{3}x$

$0 \le x \le 6$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} (3x+1)\, z \Big|_{z=0}^{z=3-\frac{3}{4}y-\frac{1}{2}x} dy\,dx$$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} (3x+1)\left(3 - \frac{3}{4}y - \frac{1}{2}x\right) dy\,dx$$

$$= \int_0^6 (3x+1)\left(\int_0^{4-\frac{2}{3}x} \left(3 - \frac{3}{4}y - \frac{1}{2}x\right) dy\right) dx$$

$$= \int_0^6 (3x+1)\left(3y - \frac{3}{8}y^2 - \frac{1}{2}xy\right)\Big|_{y=0}^{y=4-\frac{2}{3}x} dx$$

$$= \int_0^6 (3x+1)\left(12 - 6x - \frac{3}{8}\left(4 - \frac{2}{3}x\right)^2 - \cancel{2x} + \frac{1}{3}x^2\right) dx$$

$$= \int_0^6 (3x+1)\left(6 - 4x + \frac{1}{3}x^2 + 2x - \frac{1}{6}x^2\right) dx = \int_0^6 (3x+1)\left(6 - 2x + \frac{1}{6}x^2\right) dx$$

$$= \int_0^6 \left(16x - 6x^2 + \frac{1}{6}x^2 + \frac{3}{6}x^3 + 6\right) dx$$

$$= \left(16\frac{x^2}{2} - 6\frac{x^3}{3} + \frac{1}{6}\frac{x^3}{3} + \frac{1}{2}\frac{x^4}{4}\right)\Big|_0^6$$

$$= 8(36) - 2(216) + \frac{1}{18}(216) + \frac{1}{8}(1296) + 36 = \boxed{66}$$

~~8. Use Stokes Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ if $\vec{F} = <2z, x, 3y>$ and C is the ellipse in which the plane $z = x$ meets the cylinder $x^2 + y^2 = 4$, oriented counterclockwise as viewed from above.~~

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S}_1 + \iint_{S_2} \vec{F}\cdot d\vec{S}_2 + \iint_{S_3} \vec{F}\cdot d\vec{S}_3 + \iint_{S_4} \vec{F}\cdot d\vec{S}_4,$$

S: $x=0$, $y=0$, $z=0$, $2x+3y+4z=12$

S_1: $x=0$, $y=y$, $z=z$. $\quad \vec{r}(y,z) = <0, y, z>$

$\vec{r}_y = <0,1,0>$

$\vec{r}_z = <0,0,1>$.

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}(1) - \vec{j}(0) + \vec{k}(0)$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S}_1 = \iint_D <0, 0, z>\cdot<1,0,0>\,dA = \boxed{0}.$$

S_2: $\begin{cases} x=x \\ y=0 \\ z=z \end{cases}$ $\quad \vec{r}(x,z) = <x, 0, z>$

$\vec{r}_x = <1,0,0>$

$\vec{r}_z = <0,0,1>$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}(0) - \vec{j}(1) + \vec{k}(0).$$

$$\iint_{S_2} \vec{F}\cdot d\vec{S}_2 = \iint_D <x^2, 0, z>\cdot<0,-1,0>\,dA = \boxed{0}.$$

S_3: $\begin{cases} x=x \\ y=y \\ z=0 \end{cases}$ $\quad \vec{r}(x,y) = <x, y, 0>$

$\vec{r}_x = <1,0,0>$

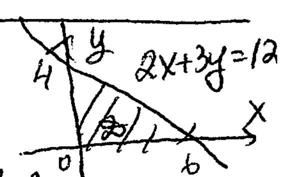
$\vec{r}_y = <0,1,0>$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(1).$$

$$\iint_{S_3} \vec{F}\cdot d\vec{S}_3 = \iint_D <x^2, xy, 0>\cdot<0,0,1>\,dA = \boxed{0}$$

S_4: $x=x$, $y=y$, $z = 3 - \frac{3}{4}y - \frac{1}{2}x$

$$\frac{\partial z}{\partial x} = -\frac{1}{2}; \quad \frac{\partial z}{\partial y} = -\frac{3}{4}.$$



$$\iint_{S_4} \vec{F}\cdot d\vec{S} = \iint_D <x^2, xy, 3 - \tfrac{3}{4}y - \tfrac{1}{2}x> <\tfrac{1}{2}, \tfrac{3}{4}, 1>\,dA$$

$$= \iint_D \left(\tfrac{1}{2}x^2 + \tfrac{3}{4}xy + 3 - \tfrac{3}{4}y - \tfrac{1}{2}x\right)dA = \int_0^6 \int_0^{4-\frac{2}{3}x} \left(\tfrac{1}{2}x^2 + \tfrac{3}{4}xy + 3 - \tfrac{3}{4}y - \tfrac{1}{2}x\right)dy\,dx$$

$$= \boxed{66}.$$

9. Use Stokes Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = < 2z, x, 3y >$ and C is the ellipse in which the plane $z = x$ meets the cylinder $x^2 + y^2 = 4$, oriented counterclockwise as viewed from above.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & x & 3y \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(3y) - \frac{\partial}{\partial z}(x)\right) - \vec{j}\left(\frac{\partial}{\partial x}(3y) - \frac{\partial}{\partial z}(2z)\right) + \vec{k}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(2z)\right)$$

$$= 3\vec{i} + 2\vec{j} + \vec{k}$$

S: $x = x$, $y = y$, $z = x$,

$\frac{\partial z}{\partial x} = 1$

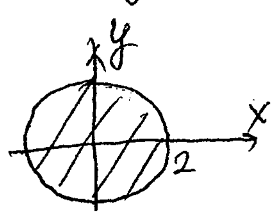
$\frac{\partial z}{\partial y} = 0$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_D < 3, 2, 1, > \cdot < -1, 0, 1 > dA$$

$$= \iint_D (-3 + 1)\, dA = -2 \iint_D dA$$

= area of a disk of radius 4

D: $x^2 + y^2 = 4$



$$= -2\,\pi(2)^2 = \boxed{-8\pi}$$