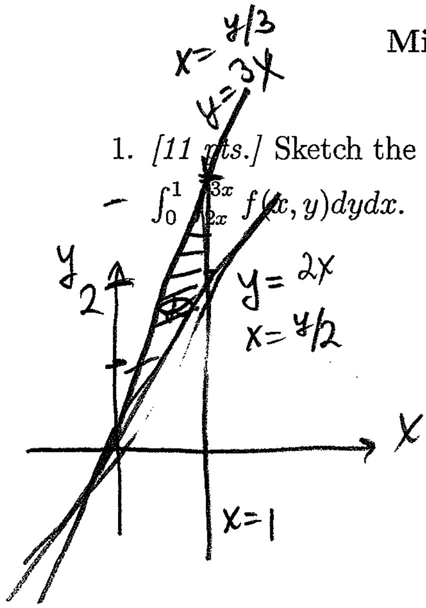
MATH 251–507, 508, 511

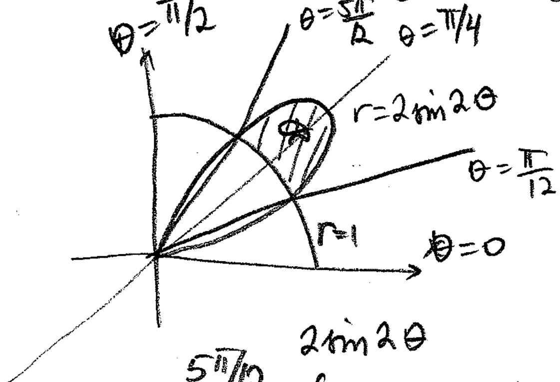
Midterm Exam II, November 18, 2010

1. *[11 pts.]* Sketch the region of integration and change the order of integration for $\int_0^1 \int_{2x}^{3x} f(x, y)\,dy\,dx$.



$$\int_0^1 \int_{2x}^{3x} f(x,y)\,dy\,dx = \int_0^2 \int_{y/3}^{y/2} f(x,y)\,dy\,dx + \int_2^3 \int_{y/3}^{1} f(x,y)\,dy\,dx$$

2. *[16 pts.]* Find the area inside one petal of the rose $r = 2\sin(2\theta)$ outside the circle $r = 1$. Sketch the region of integration.



$2\sin 2\theta = 1 \leftarrow$ intersection

$\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6},\ \frac{5\pi}{6}$

$\theta = \frac{\pi}{12},\ \frac{5\pi}{12}$

$\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$

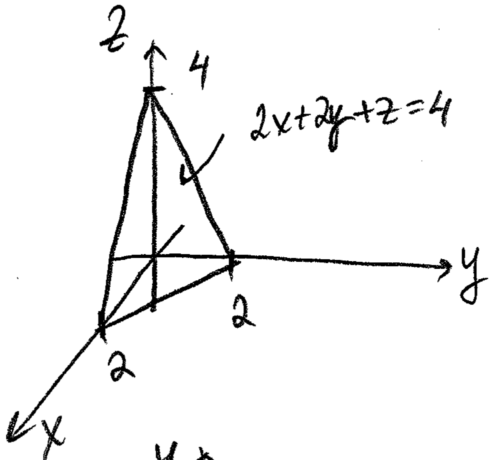
$1 \le r \le 2\sin 2\theta$

$$A = \int_{\pi/12}^{5\pi/12} \int_1^{2\sin 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12} r^2 \Big|_1^{2\sin 2\theta} dr\,d\theta$$

$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12} (4\sin^2 2\theta - 1)\,d\theta = \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2(1-\cos\cdot\theta)-1)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12} (1-\cos 4\theta)\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_{\frac{\pi}{12}}^{\frac{5\pi}{12}} = \frac{1}{2}\left(\frac{5\pi}{12} - \frac{\pi}{12} - \frac{1}{4}\sin\frac{5\pi}{3} + \frac{1}{4}\sin\frac{\pi}{3}\right) = \boxed{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$$

3. *[17 pts.]* Let E be a solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x+2y+z = 4$. Sketch the solid E and find its volume.

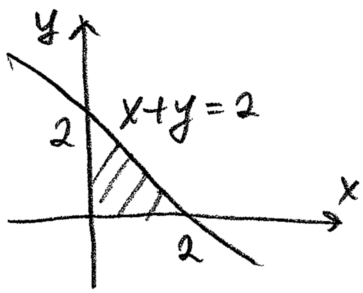


limits of integration:

$0 \le z \le 4-2x-2y$

$0 \le y \le 2-x$

$0 \le x \le 2$



$$V = \iiint_E dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} z \Big|_{z=0}^{z=4-2x-2y} dy\,dx$$

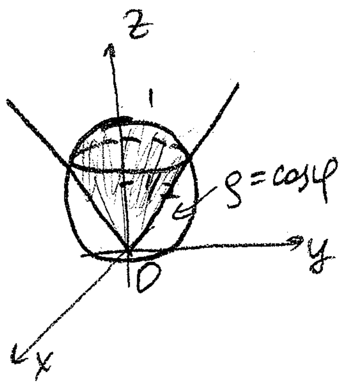
$$= \int_0^2 \int_0^{2-x} (4-2x-2y)\,dy\,dx$$

$$= \int_0^2 \left(4y - 2xy - y^2\right)\Big|_{y=0}^{y=2-x} dx = \int_0^2 \left[4(2-x) - 2x(2-x) - (2-x)^2\right] dx$$

$$= \int_0^2 \left[8 - 4x - 4x + 2x^2 - 4 + 4x - x^2\right] dx$$

$$= \int_0^2 \left[4 - 4x + x^2\right] dx = \left[4x - \frac{4x^2}{2} + \frac{x^3}{3}\right]_0^2 = 8 - 8 + \frac{8}{3} = \boxed{\frac{8}{3}}$$

4. *[14 pts.]* Consider the integral $\iiint_E y^2 dV$, where E lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$. Sketch the region of integration and convert this integral to spherical coordinates and find the limit of integration. **DO NOT EVALUATE.**



$x^2+y^2+z^2 = z$
$x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ ← sphere of radius $\frac{1}{2}$ centered at $(0,0,1/2)$

$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$

$0 \le \rho \le \cos\varphi$
$0 \le \varphi \le \pi/4$
$0 \le \theta \le 2\pi$

$x^2+y^2+z^2 = z$ in spherical coord.

$\rho^2 = \rho\cos\varphi$
$\rho = \cos\varphi$

$(z)^2 = (\sqrt{x^2+y^2})^2$
$z^2 = x^2+y^2$
$z^2+z^2 = x^2+y^2+z^2$
$2\rho^2\cos^2\varphi = \rho^2$
$\cos^2\varphi = \frac{1}{2}, \ \varphi = \pi/4.$

$$\boxed{\iiint_E y^2 dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2\sin^2\theta\sin^2\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

5. *[15 pts.]* A particle is moved by the force $\vec{F}(x,y) = y^2\vec{\imath} + 2xy\vec{\jmath}$ along the curve $\vec{r}(t) = (2+t)\vec{\imath} + t\vec{\jmath}$ from the point (2,0) to the point (3,1). What is the work done by the force?

$W = \int_C \vec{F}\cdot d\vec{r}$

$P(x,y) = y^2 \qquad Q(x,y) = 2xy.$

$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 2y \longrightarrow \vec{F}$ is conservative.

$\int_C \vec{F}\cdot d\vec{r} = f(3,1) - f(2,0)$

where $\nabla f = \vec{F}$

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$\begin{cases} \frac{\partial f}{\partial x} = y^2 \\ \frac{\partial f}{\partial y} = 2xy \end{cases}$

$f(x,y) = \int \frac{\partial f}{\partial x} dx = \int y^2 dx = y^2 x + g(y)$

$\frac{\partial f}{\partial y} = 2xy + g'(y) = 2xy$

$g'(y) = 0 \rightarrow g(y) = K$ - constant

$f(x,y) = y^2 x + K$

$W = \int_C \vec{F}\cdot d\vec{r} = (3)(1)^2 + K - (2)^2(0) - K = \boxed{3}$.

6. *[15 pts.]* For the vector field $\vec{F}(x,y,z) = x\vec{\imath} + e^y \sin z\vec{\jmath} + e^y \cos z\vec{k}$ find its curl and divergence. If the field $\vec{F}$ is conservative, find the function f such that $\nabla f = \vec{F}$.

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(e^y \sin z) + \frac{\partial}{\partial z}(e^y \cos z)$$

$$= 1 + e^y \sin z - e^y \sin z = \boxed{1}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & e^y \sin z & e^y \cos z \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(e^y \cos z) - \frac{\partial}{\partial z}(e^y \sin z)\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}(e^y \cos z) - \frac{\partial}{\partial z}(x)\right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(e^y \sin z) - \frac{\partial}{\partial y}(x)\right) = \boxed{\vec{0}}$$ — $\vec{F}$ is conservative.

$$\nabla f = \vec{F}$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\begin{cases} f_x = x \\ f_y = e^y \sin z \\ f_z = e^y \cos z \end{cases}$$

$$f(x,y,z) = \int f_x\, dx = \int x\, dx = \frac{x^2}{2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y \sin z$$

$$g(y,z) = \int e^y \sin z\, dy = e^y \sin z + h(z)$$

$$f(x,y,z) = \frac{x^2}{2} + e^y \sin z + h(z).$$

$$\frac{\partial f}{\partial z} = e^y \cos z + h'(z) = e^y \cos z$$

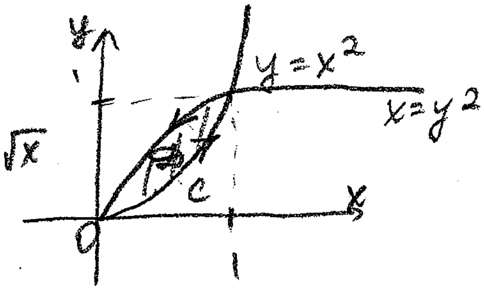
$$h'(z) = 0$$

$h(z) = K$ – constant

$$\boxed{f(x,y,z) = \frac{x^2}{2} + e^y \sin z + K}$$

7. *[12 pts.]* Use Green's Theorem to evaluate $\int_C (e^{\sqrt{x}} - xy^2)dx + (5 + \cos(y^3))dy$, where C is the boundary of the region bounded by the parabolas $y = x^2$ and $x = y^2$ with positive orientation. Sketch the region bounded by the curve C and show the orientation of C.

$0 \le x \le 1$
$x^2 \le y \le \sqrt{x}$



$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$P = e^{\sqrt{x}} - xy^2 \; ; \; \frac{\partial P}{\partial y} = -2xy$$

$$Q = 5 + \cos(y^3) \; ; \; \frac{\partial Q}{\partial x} = 0.$$

$$\int_C (e^{\sqrt{x}} - xy^2)dx + (5 + \cos(y^3))dy = \iint_D (-2xy)\,dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (-2xy)\,dy\,dx = -\int_0^1 x\,y^2\Big|_{y=x^2}^{y=\sqrt{x}} dx = -\int_0^1 x(x - x^4)\,dx$$

$$= -\int_0^1 (x^2 - x^5)\,dx = -\left(\frac{x^3}{3} - \frac{x^6}{6}\right)\Big|_0^1 = -\left(\frac{1}{3} - \frac{1}{6}\right) = \boxed{-\frac{1}{6}}$$

Bonus Problem (*[10 pts]*, **no partial credit**).

Find div $\left(\frac{\vec{r}}{|\vec{r}|}\right)$, if $\vec{r} = < x, y, z >$.

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\frac{\vec{r}}{|\vec{r}|} = \left< \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right>$$

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\sqrt{x^2+y^2+z^2} - x \cdot \frac{x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

similarly

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{div}\left(\frac{\vec{r}}{|\vec{r}|}\right) = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}} = \boxed{\frac{2}{|\vec{r}|}}$$