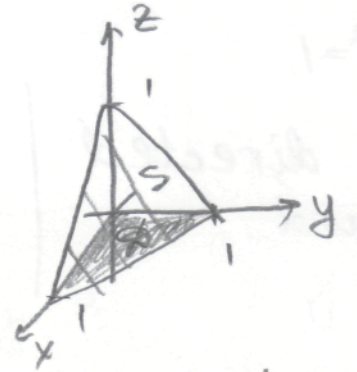
MATH 251, Section 506,507,508 Quiz#11 (Sections 14.6 – 14.9)
Tuesday, Nov. 26, 2013 Dr. M. Vorobets
Due Tuesday, Dec. 3, 2013 at the beginning of class.

NAME (print): key

No credit for unsupported answers will be given. Clearly indicate your final answer. Staple all the sheets.

1. *[3 pts.]* Evaluate $\iint_S xz\,dS$, where S is the triangle with vertices (1,0,0), (0,1,0), and (0,0,1).



Equation of S: $x+y+z=1$

$$\iint_S xz\,dS = \iint_D xz\,|\vec{n}|\,dA$$

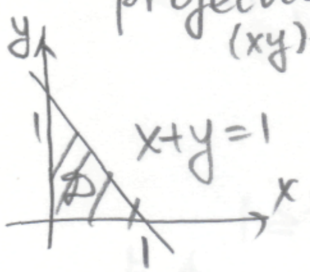
$$\vec{n} = \pm\langle z_x, z_y, -1\rangle$$

$$z = 1-x-y$$

$$\vec{n} = \pm\langle -1,-1,-1\rangle$$

$$|\vec{n}| = \sqrt{3}$$

projection onto (xy) plane



$0 \le x \le 1$

$0 \le y \le 1-x$

$$\iint_S xz\,dS = \sqrt{3}\iint_D x\overbrace{(1-x-y)}^{z}\,dA$$

$$= \sqrt{3}\int_0^1\int_0^{1-x}(x - x^2 - xy)\,dy\,dx$$

$$= \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_{y=0}^{y=1-x} dx = \sqrt{3}\int_0^1 \left(x(1-x) - x^2(1-x) - \frac{x}{2}(1-x)^2\right)dx$$

$$= \sqrt{3}\int_0^1\left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)dx = \sqrt{3}\,\frac{x^2}{4}\Big]_0^1 - \sqrt{3}\,\frac{x^3}{3}\Big]_0^1 + \sqrt{3}\,\frac{x^4}{8}\Big]_0^1$$

$$= \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right)\sqrt{3} = \boxed{\frac{\sqrt{3}}{24}}$$

2. *[4 pts.]* Use the Stoke's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x, y, z) = \left\langle x^2 y, \frac{x^3}{3}, xy \right\rangle$$

and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

$$\oint \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_D \operatorname{curl} \vec{F} \cdot \vec{n} \, dA$$

S is the portion of $z = y^2 - x^2$ inside the cylinder $x^2 + y^2 = 1$.

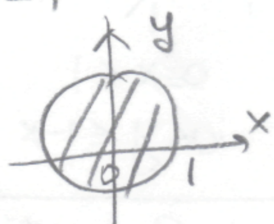
$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle = -\langle -2x, 2y, -1 \rangle = \langle 2x, -2y, 1 \rangle$$

$\vec{n}$ is directed upward

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 y & x^3/3 & xy \end{vmatrix} = \langle x, -y, 0 \rangle$$

$$\operatorname{curl} \vec{F} \cdot \vec{n} = 2x^2 + 2y^2$$

Parameter domain D: $x^2 + y^2 \le 1$



$$\oint \vec{F} \cdot d\vec{r} = \iint_D (2x^2 + 2y^2) \, dA$$

Polar coord. $x = r\cos\theta$, $y = r\sin\theta$; $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$= 2\int_0^{2\pi} \int_0^1 r^2 \, r \, dr \, d\theta = 2(2\pi) \left. \frac{r^4}{4} \right]_0^1$$

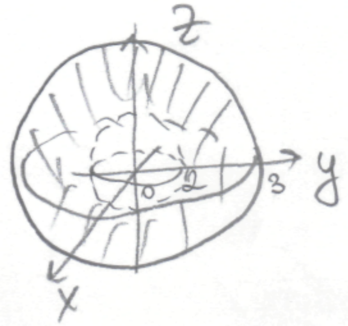
$$= \boxed{\pi}$$

3. *[4 pts.]* Use the Divergence Theorem to calculate the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3 + yz, x^2y, xy^2 \rangle$$

across the surface S, where S is the surface of the solid bounded by spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$.

$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = 4 \iiint_E x^2\, dV$$



$$\operatorname{div} \vec{F} = 3x^2 + x^2 = 4x^2$$

spherical coordinates:

$$\begin{cases} x = \rho \cos\theta \sin\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\varphi \end{cases}$$

$$2 \le \rho \le 3$$
$$0 \le \varphi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\text{flux} = 4\int_0^{2\pi}\int_0^{\pi}\int_2^{3} \rho^2\cos^2\theta \sin^2\varphi \; \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 4 \int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta \int_0^{\pi} \sin^3\varphi\, d\varphi \int_2^3 \rho^4\, d\rho$$

$$= 4 \int_0^{2\pi} \frac{1+\cancel{\cos 2\theta}^{\,0}}{2}\, d\theta \int_0^{\pi} \sin\varphi \underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi}\, d\varphi \; \frac{\rho^5}{5}\Big]_2^3$$

$$= 4(\pi)\,\frac{1}{5}(3^5 - 2^5)\int_0^{\pi} \sin\varphi(1-\cos^2\varphi)\, d\varphi$$

$u = \cos\varphi$ $\quad 0 \to 1$

$du = -\sin\varphi\, d\varphi$ $\quad \pi \to -1$

$$= -\frac{4\pi}{5}(243-32)\int_1^{-1}(1-u^2)\,du = \frac{844\pi}{5}\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1}$$

$$= \boxed{\frac{3376\pi}{15}}$$