MATH 251, Section 506, 507, 508
Thursday, Oct. 24, 2013

Quiz#7 (Sections 13.5, 13.6)
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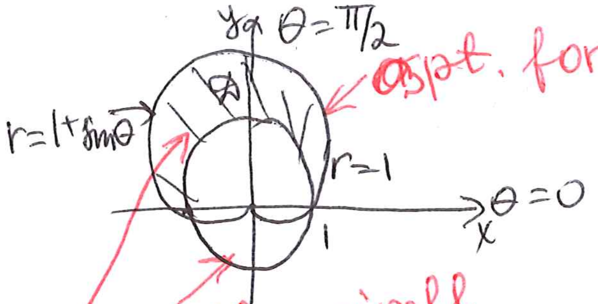
NAME (print): key

No credit for unsupported answers will be given. Clearly indicate your final answer.

1. *[10 pts.]* A lamina occupies the region inside the cardioid $r = 1 + \sin\theta$ and outside the circle $r = 1$. Find the center of mass if the density at any point is inversely proportional to its distance from the origin.



0.5pt. for cardioid

0.5pt for circle

1pt for the correct region.

$\rho = \dfrac{k}{\sqrt{x^2+y^2}} = \dfrac{k}{r}$ 1pt

intersection:

$1 + \sin\theta = 1$, $\sin\theta = 0$,
$\theta_1 = 0$, $\theta_2 = \pi$.

Polar coordinates:
$x = r\cos\theta$
$y = r\sin\theta$,

$\underset{0.5}{0} \le \theta \le \underset{0.5}{\pi}$

$\underset{0.5}{1} \le r \le \underset{0.5}{1+\sin\theta}$

$\mathscr{D}$ is symmetric about the ~~line $\theta = \pi/2$~~ y-axis,

therefore, $\bar{x} = 0$. (1pt)

$$\bar{y} = \frac{\iint\limits_{\mathscr{D}} y\rho(x,y)\,dA}{\iint\limits_{\mathscr{D}} \rho(x,y)\,dA}$$ 0.5pt

$$\iint\limits_{\mathscr{D}} \rho(x,y)\,dA = \int_0^{\pi}\int_1^{1+\sin\theta} k\,dr\,d\theta$$

$$= k\int_0^{\pi} (1+\sin\theta - 1)\,d\theta$$

$$= k(-\cos\theta)\Big|_0^{\pi}$$

$$= +\boxed{2k}$$ 1pt.

$$\iint_D y\,\rho(x,y)\,dA = \int_0^{\pi}\int_1^{1+\sin\theta} \cancel{4}\, r\sin\theta \,dr\,d\theta$$

$$= k\int_0^{\pi} \sin\theta \left.\frac{r^2}{2}\right|_{r=1}^{r=1+\sin\theta} d\theta$$

$$= \frac{k}{2}\int_0^{\pi} \sin\theta\left[(1+\sin\theta)^2 - 1\right]d\theta$$

$$= \frac{k}{2}\int_0^{\pi} \sin\theta\left[1+2\sin\theta + \sin^2\theta - 1\right]d\theta$$

$$= \frac{k}{2}\int_0^{\pi} \sin\theta\left[2\sin\theta + \sin^2\theta\right]d\theta$$

$$= \frac{k}{2}\left[\int_0^{\pi} 2\sin^2\theta\,d\theta + \int_0^{\pi}\sin^3\theta\,d\theta\right]$$

$$= \frac{k}{2}\left[\int_0^{\pi}(1-\cos 2\theta)\,d\theta + \int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta\right]$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$\theta = 0 \to u = 1$

$\theta = \pi \to u = -1$

$$= \frac{k}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} - \frac{k}{2}\int_{+1}^{-1}(1-u^2)\,du$$

$$= \frac{\pi k}{2} - \frac{k}{2}\left.\left(u - \frac{u^3}{3}\right)\right|_1^{-1}$$

$$= \frac{\pi k}{2} - \frac{k}{2}\left(-2+\frac{2}{3}\right)$$

2 pts

$$= \frac{\pi k}{2} - \frac{k}{2} = \frac{\pi k}{2} + \frac{k}{2}\cdot\frac{4}{3} = \frac{\pi k}{2} + \frac{2k}{3} = k\left(\frac{\pi}{2}+\frac{2}{3}\right)$$

0.5

$$\bar{y} = \frac{\pi/2 + 2/3}{2} = \boxed{\frac{\pi}{4}+\frac{1}{3}} \qquad \boxed{(\bar{x},\bar{y}) = \left(0, \frac{\pi}{4}+\frac{1}{3}\right)}$$