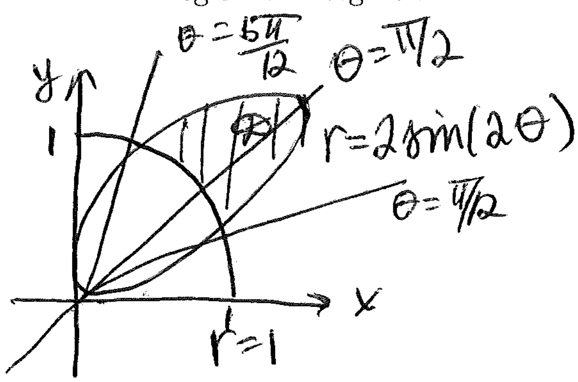
26. Find the area inside one petal of the rose $r = 2\sin(2\theta)$ outside the circle $r = 1$. Sketch the region of integration.



intersection:

$2\sin 2\theta = 1$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$$\frac{\pi}{12} \le \theta \le \frac{5\pi}{12}$$

$$1 \le r \le 2\sin(2\theta)$$

$$A = \iint_D dA = \int_{\pi/12}^{5\pi/12} \int_1^{2\sin(2\theta)} r\,dr\,d\theta = \int_{\pi/12}^{5\pi/12} \left[\frac{r^2}{2} \Big|_1^{2\sin 2\theta} \right] d\theta$$

$$= \int_{\pi/12}^{5\pi/12} \left(2\sin^2 2\theta - \frac{1}{2}\right) d\theta = \int_{\pi/12}^{5\pi/12} [1 - \cos 4\theta]\,d\theta - \frac{1}{2}\theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \left[\theta - \frac{1}{4}\sin 4\theta\right]\Big|_{\pi/12}^{5\pi/12} - \frac{1}{2}\theta \Big|_{\pi/12}^{5\pi/12}$$

$$= \frac{1}{2}\,\frac{4\pi}{12} - \frac{1}{4}\sin\left[\frac{20\pi}{12}\right] + \frac{1}{4}\sin\left(\frac{4\pi}{12}\right)$$

$$= \frac{2\pi}{12} - \frac{1}{4}\sin\frac{5\pi}{3} + \frac{1}{4}\sin\frac{\pi}{3}$$

$$= \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}$$

41. Find an equation of the tangent plane to the surface given by parametric equations $x = u^2$, $y = u - v^2$, $z = v^2$, at the point (1,0,1).

$$\vec{r}(u,v) = \langle u^2, u - v^2, v^2 \rangle$$

$$\vec{r}_u = \langle 2u, 1, 0 \rangle$$

$$\vec{r}_v = \langle 0, -2v, 2v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 1 & 0 \\ 0 & -2v & 2v \end{vmatrix} = \langle +2v, -4uv, -4vu \rangle$$

$$(1,0,1) = \vec{r}(1,1)$$

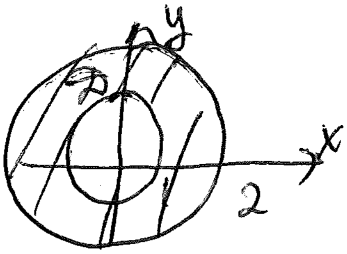
$$\left[\vec{r}_u \times \vec{r}_v\right](1,1) = \langle 2, -4, -4 \rangle$$

Tangent plane:

$$\boxed{2(x-1) - 4(y-0) - 4(z-1) = 0}$$

42. Find the area of the hyperbolic paraboloid $z = x^2 - y^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$A = \iint_S dS = \iint_D |\vec{n}|\, dA$$



$$z = x^2 - y^2$$

$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$$

$$= \pm \langle 2x, -2y, -1 \rangle$$

$$|\vec{n}| = \sqrt{1 + 4x^2 + 4y^2}.$$

$$A = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

Polar coordinates:

$x = r\cos\theta$ $\quad 1 \le r \le 2$

$y = r\sin\theta,$ $\quad 0 \le \theta \le 2\pi$

$$A = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\,dr\,d\theta = 2\pi \int_1^2 r\sqrt{1+4r^2}\, dr$$

$1 + 4r^2 = u$ $\quad 1 \to 5$

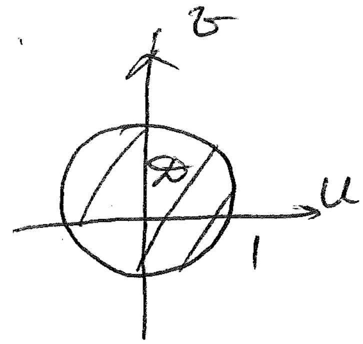
$du = 8r\,dr$ $\quad 2 \to 17$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_5^{17} = \boxed{\frac{\pi}{16}\left[17^{3/2} - 5^{3/2}\right]}$$

43. Find the area of the surface with parametric equations $x = uv$, $y = u+v$, $z = u-v$, $u^2+v^2 \leq 1$.

$$A = \iint_S dS = \iint_{\mathcal{D}} |\vec{n}|\, dA$$

$$\mathcal{D}: u^2+v^2 \leq 1$$



$$\vec{n} = \pm\, \vec{r}_u \times \vec{r}_v$$

$$\vec{r} = \langle uv, u+v, u-v \rangle$$

$$\vec{r}_u = \langle v, 1, 1 \rangle, \quad \vec{r}_v = \langle u, 1, -1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = \langle -2, u+v, v-u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4+(u+v)^2+(v-u)^2}$$

$$= \sqrt{4+u^2+2uv+v^2+v^2-2uv+u^2}$$

$$= \sqrt{4+2u^2+2v^2}$$

$$A = \iint_{\mathcal{D}} \sqrt{4+2u^2+2v^2}\, dA$$

Polar coordinates: $u = r\cos\theta$, $v = r\sin\theta$ $\quad 0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

$$A = \int_0^{2\pi}\int_0^1 r\sqrt{4+2r^2}\, dr\, d\theta = 2\pi \int_0^1 r\sqrt{4+2r^2}\, dr$$

$u = 4+2r^2 \quad 1 \to 6$

$du = 4r\,dr \quad 0 \to 4$

$$= \frac{\pi}{2}\int_4^6 \sqrt{u}\, du = \frac{\pi}{2}\cdot\frac{2}{3}u^{3/2}\Big]_4^6$$

$$= \boxed{\frac{\pi}{3}\left[6^{3/2} - 4^{3/2}\right]}$$

44. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \le z \le 4$ if its density function is $\rho(x, y, z) = 10 - z$.

$$m = \iint_S \rho(x,y,z)\,dS = \iint_D \rho(x,y,z(x,y))\,|\vec{n}|\,dA$$

$$S: \quad x = x, \quad y = y, \quad z = \sqrt{x^2+y^2}$$

$$D: \ 1 \le \sqrt{x^2+y^2} \le 4$$

$$1 \le x^2+y^2 \le 16$$

$$\vec{n} = \pm\langle z_x, z_y, -1\rangle = \pm\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right\rangle$$

$$|\vec{n}| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2}$$

$$m = \iint_D \left(10 - \overbrace{\sqrt{x^2+y^2}}^{z}\right)\sqrt{2}\,dA$$

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $1 \le r \le 4$, $0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\int_1^4 \sqrt{2}(10-r)\,r\,dr\,d\theta = 2\pi\sqrt{2}\int_1^4 (10r - r^2)\,dr\,d\theta$$

$$= 2\sqrt{2}\pi\left(5r^2 - \frac{r^3}{3}\right)\Big|_1^4$$

$$= 2\sqrt{2}\pi\left(5(15) - \frac{63}{3}\right)$$

$$= \boxed{108\sqrt{2}\pi}$$

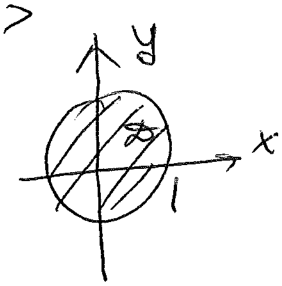
45. Evaluate $\iint_S yz\,dS$ if S is the part of the plane $z = y+3$ that lies inside the cylinder $x^2+y^2 = 1$.

$$\iint_S yz\,dS = \iint_D yz\,|\vec{n}|\,dA$$

$$z = y+3$$

$$\vec{n} = \pm \langle z_x, z_y, -1\rangle = \pm\langle 0, 1, -1\rangle$$

$$|\vec{n}| = \sqrt{2}$$



Parameter domain D: $x^2+y^2 \le 1$

$$\iint_S yz\,dS = \iint_D y\overbrace{(y+3)}^{z}\sqrt{2}\,dA$$

Polar coord.

$x = r\cos\theta \quad 0 \le r \le 1$

$y = r\sin\theta \quad 0 \le \theta \le 2\pi$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r\cos\theta(r\cos\theta+3)\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta + 3r^2\cos\theta\right] dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{1}{4}\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}} + \cos\theta\right] d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{1}{8} + \overbrace{\cancel{\frac{1}{8}\cos 2\theta}}^{0} + \overbrace{\cancel{\cos\theta}}^{0}\right] d\theta$$

$$= \boxed{\frac{\sqrt{2}\,\pi}{4}}$$