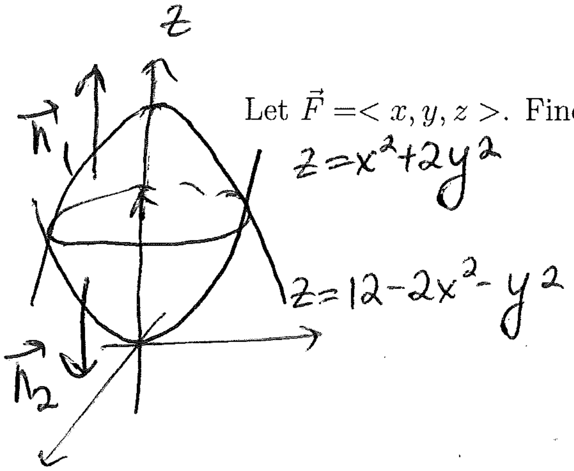
46. Let T be the solid bounded by the paraboloids

$$z = x^2 + 2y^2, \text{ and } z = 12 - 2x^2 - y^2.$$

Let $\vec{F} = < x, y, z >$. Find the outward flux of $\vec{F}$ across the boundary surface of T.



$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} + \iint_{S_2}$$

S_1: $z = x^2 + 2y^2$

$\vec{n}_2 = \pm < 2x, 4y, -1 >$ ($\vec{n}_2$ is directed downward)

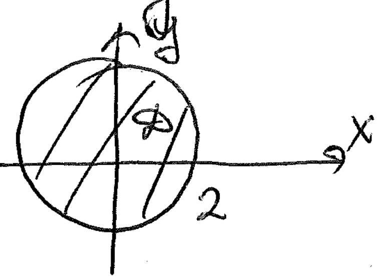
$= < 2x, 4y, -1 >$

Parameter domain:

$x^2 + 2y^2 = 12 - 2x^2 - y^2$

$3x^2 + 3y^2 = 12$

$x^2 + y^2 = 4$



$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n} \, dA$$

$$= \iint_D < x, y, \overbrace{x^2 + 2y^2}^{z} > \cdot < 2x, 4y, -1 > dA$$

$$= \iint_D (x^2 + 2y^2) \, dA$$

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_0^2 (r^2\cos^2\theta + 2r^2\sin^2\theta) \, r\,dr\,d\theta = \int_0^{2\pi} \int_0^2 (r^3 + \cancel{2}r^3\sin^2\theta) \, dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3 (1 + \sin^2\theta) \, dr\,d\theta = \int_0^{2\pi} 4\left(1 + \frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{2\pi} 4\left(\frac{3}{2} - \cancel{\frac{\cos 2\theta}{2}}^{\,0}\right) d\theta = 6(2\pi) = \boxed{12\pi}$$

S_2: $z = 12 - 2x^2 - y^2$

$$\vec{n}_1 = \pm \langle z_x, z_y, -1 \rangle = - \langle -4x, -2y, -1 \rangle$$

$$= \langle 4x, 2y, 1 \rangle$$

$\vec{n}_1$ is directed upward.

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{D} \vec{F} \cdot \vec{n}\, dA = \iint_{D} \langle x, y, 12 - 2x^2 - y^2 \rangle \cdot \langle 4x, 2y, 1 \rangle\, dA$$

$$= \iint_{D} (2x^2 + y^2 + 12)\, dA$$

Polar coord.	
$x = r\cos\theta$	$0 \le r \le 2$
$y = r\sin\theta$	$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_0^2 [r^2 + r^2 \cos^2\theta + 12]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 [r^3 (1 + \cos^2\theta) + 12r]\, dr\, d\theta$$

$$= \int_0^{2\pi} [4(1 + \cos^2\theta) + 24]\, d\theta$$

$$= \int_0^{2\pi} \left[4\left(1 + \frac{1 + \cos 2\theta}{2}\right) + 24\right] d\theta$$

$$= \int_0^{2\pi} \left[4\left[\frac{3}{2} + \cancelto{0}{\frac{\cos 2\theta}{2}}\right] + 24\right] d\theta$$

$$= 30(2\pi) = \boxed{60\pi}$$

$$\iint_S = 60\pi + 12\pi = \boxed{72\pi}$$

7'

OR Divergence Theorem:

$$\text{flux} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$\operatorname{div} \vec{F} = 3$,

cylindrical coord.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$12 - 2x^2 - y^2 \geq z \geq x^2 + 2y^2$
$12 - r^2 - r^2\cos^2\theta \geq z \geq r^2 + r^2\sin^2\theta$

$0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

$$\text{flux} = 3\int_0^{2\pi} \int_0^2 \int_{r^2 + r^2\sin^2\theta}^{12 - r^2 - r^2\cos^2\theta} r \, dz \, dr \, d\theta$$

$$= 3\int_0^{2\pi} \int_0^2 r\left[12 - r^2 - r^2\cos^2\theta - r^2 - r^2\sin^2\theta\right] dr\, d\theta$$

$$= 3\int_0^{2\pi} \int_0^2 r\left[12 - 3r^2\right] dr\, d\theta = 9(2\pi)\int_0^2 (4r - r^3)\, dr$$

$$= 18\pi\left(\frac{4r^2}{2} - \frac{r^4}{4}\right)\Big|_0^2$$

$$= 18\pi(8 - 4) = 18\pi(4) = \boxed{72\pi}$$