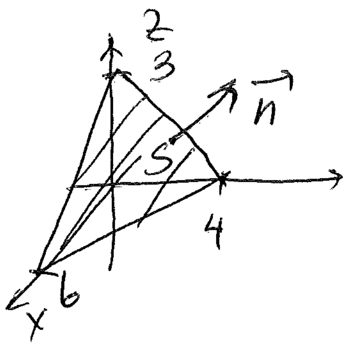
47. Verify the Divergence Theorem for $\vec{F} = \langle x^2, xy, z \rangle$ and the region E bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$.



$$\iint_S \vec{F}\cdot d\vec{S} \overset{?}{=} \iiint_E \operatorname{div}\vec{F}\, dV$$

outward flux.

LHS: S_1: $2x+3y+4z=12$

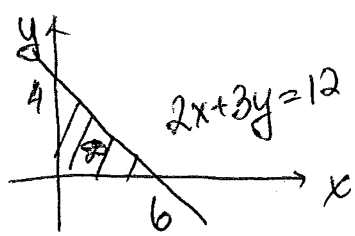
$$z = \frac{1}{4}(12-3y-2x)$$

$$\vec{n}_1 = \pm\langle z_x, z_y, -1\rangle = \ominus\left\langle -\frac{1}{2}, -\frac{3}{4}, -1\right\rangle = \left\langle \frac{1}{2}, \frac{3}{4}, 1\right\rangle$$

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{D} \underbrace{\left[\frac{1}{2}x^2 + \frac{3}{4}xy + \underbrace{\frac{1}{4}(12-3y-2x)}_{z}\right]}_{\vec{F}\cdot\vec{n}} dA$$

$$= \iint_D \left[\frac{1}{2}x^2 + \frac{3}{4}xy + 3 - \frac{3}{4}y - \frac{1}{2}x\right] dA$$

D:



$0 \le x \le 6$

$0 \le y \le 4 - \frac{2x}{3}$

$$= \int_0^6 \int_0^{4-\frac{2x}{3}} \left(\frac{1}{2}x^2 + \frac{3}{4}xy + 3 - \frac{3}{4}y - \frac{1}{2}x\right] dy\, dx$$

$$= \int_0^6 \left(\frac{1}{2}x^2 y + \frac{3}{4}x\frac{y^2}{2} + 3y - \frac{3}{4}\frac{y^2}{2} - \frac{1}{2}xy\right)\Big|_0^{4-\frac{2x}{3}} dx$$

$$= \int_0^6 \left[\frac{1}{2}x^2\left(4-\frac{2x}{3}\right) + \frac{3}{8}x\left(4-\frac{2x}{3}\right)^2 + 3\left(4-\frac{2x}{3}\right) - \frac{3}{8}\left(4-\frac{2x}{3}\right)^2 - \frac{1}{2}x\left(4-\frac{2x}{3}\right)\right] dx = \ldots = 66$$

S_2: $x=0$ (yz-plane). $\vec{n}_2 = \langle -1, 0, 0\rangle$

$$\vec{F}\cdot\vec{n}_2 = \langle 0, 0, z\rangle \cdot \langle -1, 0, 0\rangle = 0.$$

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = 0.$$

$S_3: y=0$ (xz-plane.) $\vec{n}_3 = \langle 0,-1,0 \rangle$.

$$\vec{F}\cdot\vec{n}_3 = \langle x^2, 0, z\rangle \cdot \langle 0,-1,0\rangle = 0$$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = 0$$

$S_4: z=0$ (xy-plane) $\vec{n}_4 = \langle 0,0,-1\rangle$

$$\vec{F}\cdot\vec{n}_4 = \langle x^2, xy, 0\rangle\cdot\langle 0,0,-1\rangle = 0.$$

$$\iint_{S_4} \vec{F}\cdot d\vec{S} = 0$$

$$\iint_S \vec{F}\cdot d\vec{n} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} = \boxed{66}$$

RHS: $\operatorname{div}\vec{F} = 2x + x + 1 = 3x+1$

$0 \le z \le 3 - \frac{3}{4}y - \frac{1}{2}x$

$0 \le y \le 4 - \frac{2}{3}x$

$0 \le x \le 6$

$$\iiint_E \operatorname{div}\vec{F}\, dV = \int_0^6 \int_0^{4-\frac{2}{3}x} \int_0^{3-\frac{3}{4}y-\frac{1}{2}x} (3x+1)\, dz\, dy\, dx$$

$$= \int_0^6 \int_0^{4-\frac{2}{3}x} (3x+1)\left(3 - \frac{3}{4}y - \frac{1}{2}x\right) dy\, dx$$

$$= \int_0^6 \left(9x - \frac{9}{4}xy - \frac{3}{2}x^2 + 3 - \frac{3}{4}y - \frac{1}{2}x\right) dy\, dx$$

$$= \int_0^6 (3x+1)\left(3y - \frac{3}{8}y^2 - \frac{1}{2}xy\right)\Big|_{y=0}^{y=4-\frac{2}{3}x} dx$$

$$= \int_0^6 (3x+1)\left(3\left(4-\frac{2}{3}x\right) - \frac{3}{8}\left(4-\frac{2}{3}x\right)^2 - \frac{1}{2}x\left(4-\frac{2}{3}x\right)\right) dx$$

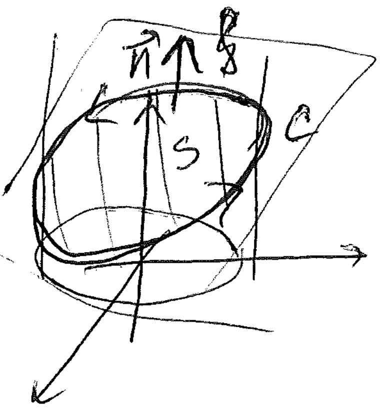
$$= \int_0^6 (3x+1)\left(6 - 2x + \frac{1}{6}x^2\right) dx$$

$$= \int_0^6 \left(16x - 6x^2 + \frac{1}{6}x^2 + \frac{3}{6}x^3 + 6\right) dx$$

$$= \left[8x^2 - 2x^3 + \frac{x^3}{18} + \frac{x^4}{8} + 6x\right]_0^6 = 8(36) - 2(216) + \frac{1}{18}(216) + \frac{1296}{8} + 36 = \boxed{66}.$$

$\boxed{LHS = RHS}$

48. Use Stokes Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ if $\vec{F} = < 2z, x, 3y >$ and C is the ellipse in which the plane $z = x$ meets the cylinder $x^2 + y^2 = 4$, oriented counterclockwise as viewed from above.

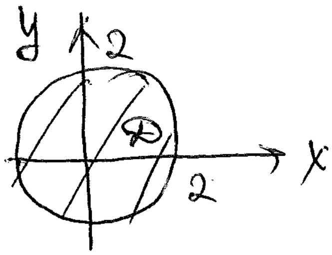


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$$= \iint_{\mathcal{D}} \text{curl}\,\vec{F}\cdot \vec{n}\, dA$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z & x & 3y \end{vmatrix} = <3, 2, 1>$$

S: $\cancel{x^2+y^2=4}$ $\cancel{z=x}$ $\quad z = x.$



$$\vec{n} = \pm < z_x, z_y, -1 >$$

$$= \pm < 1, 0, -1 >$$

$$= <-1, 0, 1>.$$

Parameter domain : $\mathcal{D} = \{x^2+y^2 \le 4\}$

$$\iint_{\mathcal{D}} \text{curl}\,\vec{F}\cdot\vec{n}\, dA = \iint_{\mathcal{D}} (-3+1)\,dA = -2\iint_{\mathcal{D}} dA$$

$$= -2(\text{area } \mathcal{D})$$

$$= -2\pi 2^2$$

$$= \boxed{-8\pi}$$