

Practice problems for Exam 1.

1. Given $\vec{a} = \langle 1, 1, 2 \rangle$ and $\vec{b} = \langle 2, -1, 0 \rangle$. Find the area of the parallelogram with adjacent sides \vec{a} and \vec{b} .

SOLUTION. $A = |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 2 & -1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 2\vec{i} + 4\vec{j} - 3\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(2)^2 + (4)^2 + (-3)^2} = \sqrt{29}$$

Thus, $A = \sqrt{29}$.

2. Find an equation of the line through the point $(1, 2, -1)$ and perpendicular to the plane

$$2x + y + z = 2$$

SOLUTION. The line is parallel to the normal vector of the plane $\vec{n} = \langle 2, 1, 1 \rangle$. Thus, symmetric equations of the line are:

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+1}{1}$$

3. Find the distance from the point $(1, -1, 2)$ to the plane

$$x + 3y + z = 7$$

SOLUTION. $D = \frac{|1 + 3(-1) + 2 - 7|}{\sqrt{(1)^2 + (3)^2 + (1)^2}} = \frac{7}{\sqrt{11}}$.

4. Find an equation of the plane that passes through the point $(-1, -3, 1)$ and contains the line $x = -1 - 2t$, $y = 4t$, $z = 2 + t$.

SOLUTION. The vector $\vec{v} = \langle -2, 4, 1 \rangle$ lies in the plane. Let $P(-1, -3, 1)$ and $Q(-1, 0, 2)$. The second vector that lies in the plane is the vector $\vec{PQ} = \langle 0, 3, 1 \rangle$. Then the normal vector to the plane

$$\vec{n} = \vec{v} \times \vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 4 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 4 \\ 0 & 3 \end{vmatrix} = \vec{i} + 2\vec{j} - 6\vec{k}$$

Thus, an equation of the plane is

$$1(x+1) + 2(y+3) - 6(z-1) = 0$$

5. Find parametric equations of the line of intersection of the planes $z = x + y$ and $2x - 5y - z = 1$.

SOLUTION. The direction vector for the line of intersection is $\vec{v} = \vec{n}_1 \times \vec{n}_2$, where $\vec{n}_1 = \langle 1, 1, -1 \rangle$ is the normal vector for the first plane and $\vec{n}_2 = \langle 2, -5, -1 \rangle$ is the normal vector for the second plane.

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -5 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -1 \\ -5 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} =$$

$$\vec{i}(-1-5) - \vec{j}(-1+2) + \vec{k}(-5-2) = -6\vec{i} - \vec{j} - 7\vec{k}$$

. To find a point on the line of intersection, set one of the variables equal to a constant, say $y = 0$. Then the equations of the planes reduce to $x - z = 0$ and $2x - z = 1$. Solving this two equations gives $x = z = 1$. So a point on a line of intersection is $(1, 0, 1)$. The parametric equations for the line are

$$\begin{aligned} x &= 1 - 6t \\ y &= -t \\ z &= 1 - 7t \end{aligned}$$

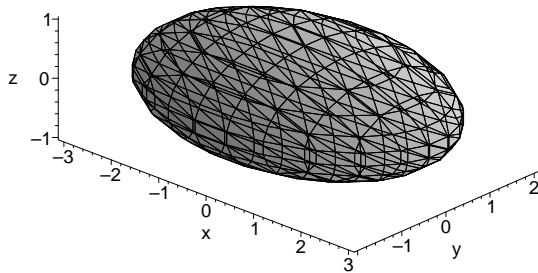
6. Are the lines $x = -1 + 4t, y = 3 + t, z = 1$ and $x = 13 - 8s, y = 1 - 2s, z = 2$ parallel, skew or intersecting? If they intersect, find the point of intersection.

SOLUTION. The direction vector for the first line is $\vec{v}_1 = \langle 4, 1, 0 \rangle$, the second line is parallel to the vector $\vec{v}_2 = \langle -8, -2, 0 \rangle$. Since $\vec{v}_2 = -2\vec{v}_1$, vectors \vec{v}_1 and \vec{v}_2 are parallel. Thus, the lines are parallel.

7. Identify and roughly sketch the following surfaces. Find traces in the planes $x = k, y = k, z = k$

(a) $4x^2 + 9y^2 + 36z^2 = 36$

SOLUTION. $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ - ellipsoid



Traces

in $x = k$: $\frac{y^2}{4(1 - \frac{k^2}{9})} + \frac{z^2}{1 - \frac{k^2}{9}} = 1$ - ellipse

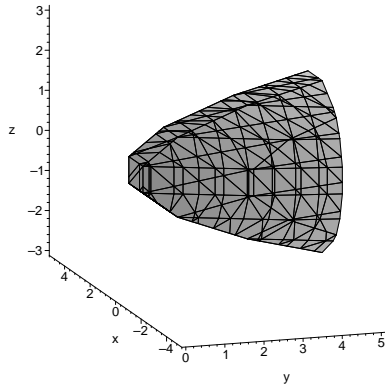
in $y = k$: $\frac{x^2}{9(1 - \frac{k^2}{4})} + \frac{z^2}{1 - \frac{k^2}{4}} = 1$ - ellipse

in $z = k$: $\frac{x^2}{9(1 - k^2)} + \frac{y^2}{4(1 - k^2)} = 1$ - ellipse

(b) $y = x^2 + z^2$

An equation $y = x^2 + z^2$ defines the elliptic paraboloid with axis the y -axis.

SOLUTION.



Traces

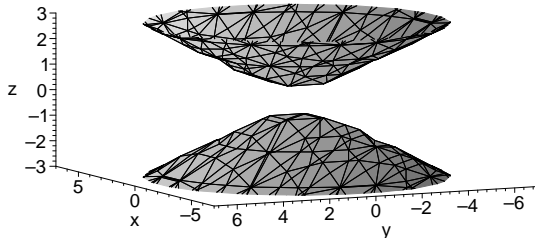
in $x = k$: $y = z^2 + k^2$ - parabola

in $y = k$: $x^2 + z^2 = k$ - circle

in $z = k$: $y = x^2 + k^2$ - parabola

(c) $4z^2 - x^2 - y^2 = 1$

SOLUTION. An equation $4z^2 - x^2 - y^2 = 1$ defines the hyperboloid on two sheets with axis the z -axis



Traces

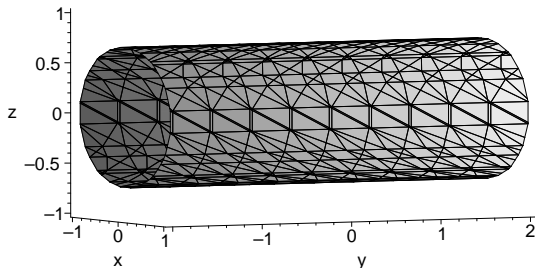
in $x = k$: $\frac{z^2}{1/4(1 - k^2)} - \frac{y^2}{1 - k^2} = 1$ - hyperbola

in $y = k$: $\frac{z^2}{1/4(1 - k^2)} - \frac{x^2}{1 - k^2} = 1$ - hyperbola

in $z = k$ ($|k| > 1/2$): $x^2 + y^2 = 4k^2 - 1$ - circle

(d) $x^2 + 2z^2 = 1$

SOLUTION. An equation $x^2 + 2z^2 = 1$ defines the elliptic cylinder with axis y -axis.



Traces

in $x = k$: $z = \sqrt{\frac{1 - k^2}{2}}$, $z = -\sqrt{\frac{1 - k^2}{2}}$ - lines

in $z = k$: $x = \sqrt{1 - 2k^2}$, $x = -\sqrt{1 - 2k^2}$ - lines

8. Find

$$\lim_{t \rightarrow 1} \left(\sqrt{t + 3\vec{i}} + \frac{t - 1}{t^2 - 1} \vec{j} + \frac{\tan t}{t} \vec{k} \right)$$

SOLUTION.

$$\lim_{t \rightarrow 1} \left(\sqrt{t + 3\vec{i}} + \frac{t - 1}{t^2 - 1} \vec{j} + \frac{\tan t}{t} \vec{k} \right) = \lim_{t \rightarrow 1} \sqrt{t + 3\vec{i}} + \lim_{t \rightarrow 1} \frac{t - 1}{t^2 - 1} \vec{j} + \lim_{t \rightarrow 1} \frac{\tan t}{t} \vec{k} =$$

$$\sqrt{4\vec{i}} + \lim_{t \rightarrow 1} \frac{t - 1}{(t - 1)(t + 1)} \vec{j} + \tan(1) \vec{k} = 2\vec{i} + \frac{1}{2} \vec{j} + \tan(1) \vec{k}$$

9. Find the unit tangent vector $\vec{T}(t)$ for the vector function $\vec{r}(t) = \langle t, 2 \sin t, 3 \cos t \rangle$.

SOLUTION. The tangent vector $\vec{r}'(t) = \langle 1, 2 \cos t, -3 \sin t \rangle$,

$$|\vec{r}'(t)| = \sqrt{1 + 4 \cos^2 t + 9 \sin^2 t} = \sqrt{1 + 4 \cos^2 t + 4 \sin^2 t + 5 \sin^2 t} = \sqrt{5 + 5 \sin^2 t}$$

. The unit tangent vector

$$\vec{T}(t) = \frac{1}{|\vec{r}'(t)|} \vec{r}'(t) = \frac{1}{\sqrt{5 + 5 \sin^2 t}} \langle 1, 2 \cos t, -3 \sin t \rangle$$

10. Evaluate

$$\int_1^4 \left(\sqrt{t} \vec{i} + t e^{-t} \vec{j} + \frac{1}{t^2} \vec{k} \right) dt$$

SOLUTION.

$$\int_1^4 \left(\sqrt{t} \vec{i} + t e^{-t} \vec{j} + \frac{1}{t^2} \vec{k} \right) dt = \left(\int_1^4 \sqrt{t} dt \right) \vec{i} + \left(\int_1^4 t e^{-t} dt \right) \vec{j} + \left(\int_1^4 \frac{1}{t^2} dt \right) \vec{k}$$

$$\int_1^4 \sqrt{t} dt = \left. \frac{t^{3/2}}{3/2} \right|_1^4 = \frac{2}{3} [4^{3/2} - 1] = \frac{2}{3} (8 - 1) = \frac{14}{3}$$

$$\int_1^4 t e^{-t} dt = \left| \begin{array}{l} u = t \quad u' = 1 \\ v' = e^{-t} \quad v = -e^{-t} \end{array} \right| = -t e^{-t} \Big|_1^4 + \int_1^4 e^{-t} dt = -4e^{-4} + e^{-1} - e^{-t} \Big|_1^4 =$$

$$-4e^{-4} + e^{-1} - e^{-4} + e^{-1} = -5e^{-4} + 2e^{-1}$$

$$\int_1^4 \frac{1}{t^2} dt = \left. \frac{1}{t} \right|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

Thus,

$$\int_1^4 \left(\sqrt{t} \vec{i} + t e^{-t} \vec{j} + \frac{1}{t^2} \vec{k} \right) dt = \frac{14}{3} \vec{i} + (-5e^{-4} + 2e^{-1}) \vec{j} + \frac{3}{4} \vec{k}$$

11. Find the length of the curve given by the vector function $\vec{r}(t) = \cos^3 t \vec{i} + \sin^3 t \vec{j} + \cos(2t) \vec{k}$, $0 \leq t \leq \frac{\pi}{2}$.

SOLUTION. $\vec{r}'(t) = -3 \cos^2 t \sin t \vec{i} + 3 \sin^2 t \cos t \vec{j} - 2 \sin(2t) \vec{k}$

$$|\vec{r}'(t)| = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t + 4 \sin^2(2t)}$$

Recall that $\sin(2t) = 2 \sin t \cos t$, then $\sin^2(2t) = 4 \sin^2 t \cos^2 t$ and

$$|\vec{r}'(t)| = \sqrt{\sin^2 t \cos^2 t (9 \sin^2 t + 9 \cos^2 t + 16)} = \sin t \cos t \sqrt{25} = 5 \sin t \cos t = \frac{5}{2} \sin(2t)$$

Then the length of the curve

$$L = \int_0^{\pi/2} \frac{5}{2} \sin(2t) dt = -\frac{5}{4} \cos(2t) \Big|_0^{\pi/2} = \frac{5}{2}$$

12. Find the curvature of the curve $\vec{r}(t) = \langle 2t^3, -3t^2, 6t \rangle$.

SOLUTION.

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

$$\vec{r}'(t) = \langle 6t^2, -6t, 6 \rangle = 6 \langle t^2, -t, 1 \rangle, \quad |\vec{r}'(t)| = 6\sqrt{1+t^2+t^4}$$

$$\vec{r}''(t) = \langle 12t, -6, 0 \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6t^2 & -6t & 6 \\ 12t & -6 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} -6t & 6 \\ -6 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 6t^2 & 6 \\ 12t & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 6t^2 & -6t \\ 12t & -6 \end{vmatrix} =$$

$$\vec{i}(36) - \vec{j}(-72t) + \vec{k}(-36t^2 + 72t^2) = \langle 36, 72t, 36t^2 \rangle = 36 \langle 1, 2t, t^2 \rangle$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = 36\sqrt{1+4t^2+t^4}$$

Thus,

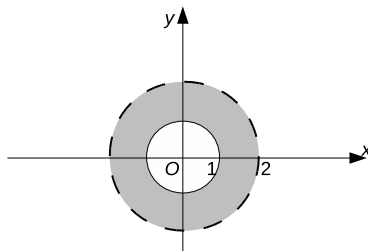
$$\kappa(t) = \frac{36\sqrt{1+4t^2+t^4}}{(6\sqrt{1+t^2+t^4})^3} = \frac{\sqrt{1+4t^2+t^4}}{6(1+t^2+t^4)^{3/2}}$$

13. Sketch the domain of the function

$$f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)$$

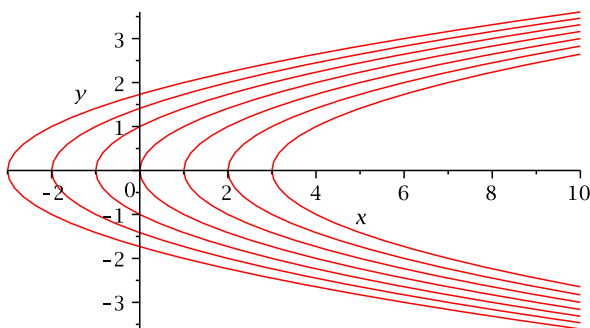
SOLUTION. The expression for f makes sense if $x^2 + y^2 - 1 \geq 0$ and $4 - x^2 - y^2 > 0$. Thus, the domain of the function f is

$$D(f) = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 < 4\}$$



14. Find the level curves of the function $z = x - y^2$.

SOLUTION. An equation for the level curves is $k = x - y^2$ or $y^2 = x - k$. It defines the family of parabolas.



15. Find f_{xyz} if $f(x, y, z) = e^{xyz}$.

SOLUTION. $f_x = e^{xyz}(xyz)'_x = yze^{xyz}$

$$f_{xy} = (yze^{xyz})'_y = ze^{xyz} + yze^{xyz}(xyz)'_y = ze^{xyz} + xyz^2e^{xyz}$$

$$f_{xyz} = (ze^{xyz} + xyz^2e^{xyz})'_z = e^{xyz} + ze^{xyz}(xyz)'_z + 2xyze^{xyz} + xyz^2e^{xyz}(xyz)'_z = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2y^2z^2e^{xyz} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$$

16. The dimensions of a closed rectangular box are 80 cm, 60 cm, and 50 cm with a possible error of 0.2 cm in each dimension. Use differential to estimate the maximum error in surface area of the box.

SOLUTION. Let l , w , and h be the length, width, and height, respectively, of the box in centimeters.

$$\Delta l = \Delta w = \Delta h = 0.2$$

The surface area of the box

$$A(l, w, h) = 2(lw + lh + wh)$$

$$\Delta A = \frac{\partial A}{\partial l} \Delta l + \frac{\partial A}{\partial w} \Delta w + \frac{\partial A}{\partial h} \Delta h$$

$$\frac{\partial A}{\partial l} = 2w + 2h; \quad \frac{\partial A}{\partial l}(80, 60, 50) = 220$$

$$\frac{\partial A}{\partial w} = 2l + 2h; \quad \frac{\partial A}{\partial w}(80, 60, 50) = 260$$

$$\frac{\partial A}{\partial h} = 2l + 2w; \quad \frac{\partial A}{\partial h}(80, 60, 50) = 280$$

Thus,

$$\Delta A = (220 + 260 + 280)(0.2) = 152\text{cm}^2$$

17. Find parametric equations of the normal line and an equation of the tangent plane to the surface

$$x^3 + y^3 + z^3 = 5xyz$$

at the point $(2, 1, 1)$.

SOLUTION. Let $F(x, y, z) = x^3 + y^3 + z^3 - 5xyz$. Then

$$\nabla F(x, y, z) = \langle 3x^2 - 5yz, 3y^2 - 5xz, 3z^2 - 5xy \rangle$$

$$\nabla F(2, 1, 1) = \langle 7, -7, -7 \rangle$$

The equation of the tangent plane at $(2, 1, 1)$ is

$$7(x - 2) - 7(y - 1) - 7(z - 1) = 0$$

The parametric equations of the normal line at $(2, 1, 1)$ are

$$x = 2 + 7t, \quad y = 1 - 7t, \quad z = 1 - 7t$$

18. Given that $w(x, y) = 2 \ln(3x + 5y) + x - 2 \tan^{-1} y$, where $x = s - \cot t$, $y = s + \sin^{-1} t$. Find $\frac{\partial w}{\partial t}$.

SOLUTION.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} = \left(\frac{6}{3x + 5y} + 1 \right) \csc^2 t + \left(\frac{10}{3x + 5y} - \frac{2}{1 + y^2} \right) \frac{1}{\sqrt{1 - t^2}}$$

19. Let $f(x, y, z) = \ln(2x + 3y + 6z)$. Find a unit vector in the direction in which f decreases most rapidly at the point $P(-1, -1, 1)$ and find the derivative (rate of change) of f in this direction.

SOLUTION. The function f decreases most rapidly in the direction of the vector $-\nabla f(-1, -1, 1)$.

$$\nabla f(x, y, z) = \left\langle \frac{2}{2x + 3y + 6z}, \frac{3}{2x + 3y + 6z}, \frac{6}{2x + 3y + 6z} \right\rangle$$

$$\nabla f(-1, -1, 1) = \left\langle \frac{2}{-2 - 3 + 6}, \frac{3}{-2 - 3 + 6}, \frac{6}{-2 - 3 + 6} \right\rangle = \langle 2, 3, 4 \rangle$$

$$|\nabla f(-1, -1, 1)| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

The unit vector in the direction of the vector $\nabla f(-1, -1, 1)$ is

$$\vec{u} = \frac{1}{7} \langle 2, 3, 4 \rangle = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{4}{7} \right\rangle$$

So, the function f decreases most rapidly in the direction of the vector $-\vec{u} = \left\langle -\frac{2}{7}, -\frac{3}{7}, -\frac{4}{7} \right\rangle$

$$D_{-\vec{u}} f(x, y, z) = \nabla f(x, y, z) \cdot (-\vec{u}) = \left\langle \frac{2}{2x + 3y + 6z}, \frac{3}{2x + 3y + 6z}, \frac{6}{2x + 3y + 6z} \right\rangle \cdot \left\langle -\frac{2}{7}, -\frac{3}{7}, -\frac{4}{7} \right\rangle = -\frac{7}{2x + 3y + 6z}$$

20. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$xe^y + yz + ze^x = 0$$

SOLUTION.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

where $F(x, y, z) = xe^y + yz + ze^x$

$$\frac{\partial F}{\partial x} = e^y + ze^x$$

$$\frac{\partial F}{\partial y} = xe^y + z$$

$$\frac{\partial F}{\partial z} = y + e^x$$

$$\text{So } \frac{\partial z}{\partial x} = -\frac{e^y + ze^x}{y + e^x} \text{ and } \frac{\partial z}{\partial y} = -\frac{xe^y + z}{y + e^x}.$$

21. Find the local extrema/saddle points for

$$f(x, y) = 2x^2 + y^2 + 2xy + 2x + 2y$$

SOLUTION. We first locate the critical points:

$$f_x(x, y) = 2x + y + 1$$

$$f_y(x, y) = y + x + 1$$

Setting these derivatives equal to zero, we get the following system:

$$\begin{cases} 2x + y + 1 = 0 \\ x + y + 1 = 0 \end{cases}$$

Substitute $y = -1 - x$ from the second equation into the first equation:

$$2x + (-1 - x) + 1 = 0$$

$$x = 0, y = -1 - x = -1$$

The critical point is $(0, -1)$.

Next we calculate the second partial derivatives:

$$f_{xx}(x, y) = 4, f_{xy}(x, y) = 2, f_{yy}(x, y) = 2.$$

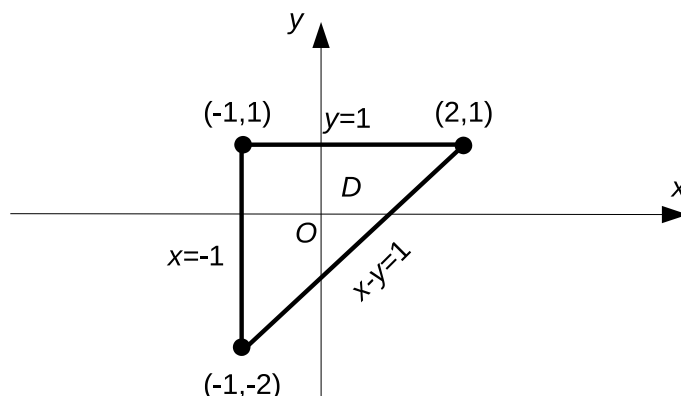
Then

$$D(x, y) = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0$$

Since $D(x, y) = 4 > 0$ and $f_{xx}(x, y) = 4 > 0$, the function f has a local minimum at the point $(0, -1)$, $f(0, -1) = -1$.

22. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ over the set D , where D is the closed triangular region with vertices $(-1, 1)$, $(2, 1)$, and $(-1, -2)$.

SOLUTION. The set D is bounded by lines $x = -1$, $y = 1$, and $x - y = 1$



First we find critical points for f :

$$f_x(x, y) = 2x + 2y = 0$$

$$f_y(x, y) = 2x + 6y = 0$$

so the only critical point is $(0, 0)$.

$$f(0, 0) = 0$$

Now we look at the values of f on the boundary of D .

$$f(-1, 1) = (-1)^2 + 2(-1)(1) + 3(1)^2 = 2$$

$$f(2, 1) = (2)^2 + 2(2)(2) + 3(1)^2 = 11$$

$$f(-1, -2) = (-1)^2 + 2(-1)(-2) + 3(-2)^2 = 17$$

$$\text{If } x = -1, \text{ then } f(-1, y) = 1 - 2y + 3y^2$$

$$f_y(-1, y) = -2 + 6y = 0, \text{ so } y = 1/3.$$

$$f(-1, 1/3) = 1/3$$

$$\text{If } y = 1, \text{ then } f(x, 1) = x^2 + 2x + 1, f_x(x, 1) = 2x + 2 = 0 \text{ and } x = -1$$

$$f(-1, 1) = 2$$

$$\text{If } x - y = 1, \text{ then } y = x - 1, \text{ and } g(x) = f(x, y) = x^2 + 2xy + 3y^3 = x^2 + 2x(x - 1) + 3(x - 1)^2 = 6x^2 - 8x - 3$$

$$g'(x) = 12x - 8 = 0, \text{ so } x = 8/12 = 2/3 \text{ and } y = 2/3 - 1 = -1/3$$

$$f(2/3, -1/3) = 1/3$$

Thus, the absolute maximum value of the function is $f(-1, -2) = 17$ and the absolute minimum value is $f(0, 0) = 0$.