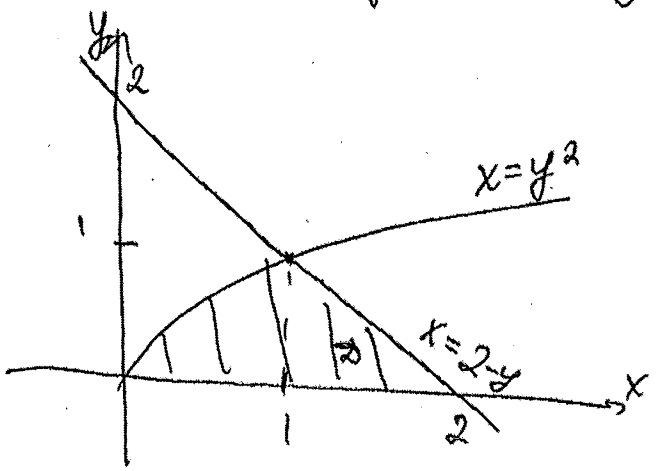
Solutions for the practice problems for Exam 2.

1. Sketch the region of integration and change the order of integration for $\int_0^1 \int_{y^2}^{2-y} f(x,y)\,dx\,dy$.

The region is bounded by

$y=0$, $y=0$

$x=y^2$, $x=2-y$



point of intersection:

$y^2=2-y$

$y^2+y-2=0$

$y_1=1$ $\quad y_2=-2<0$ not valid.

$x=1$

point of intersection is $(1,1)$.

if $0\le x\le 1$, then $0\le y\le \sqrt{x}$

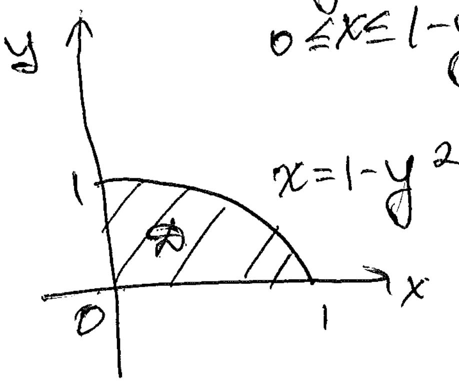
if $1\le x\le 2$, then $0\le y\le 2-x$.

$$\int_0^1\int_{y^2}^{2-y} f(x,y)\,dx\,dy = \int_0^1\int_0^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^2\int_0^{2-x} f(x,y)\,dy\,dx.$$

2. Evaluate $\iint_D (xy+2x+3y)dA$, where D is the region in the first quadrant bounded by $x = 1-y^2$, $x = 0$, $y = 0$.

$0 \le y \le 1$

$0 \le x \le 1-y^2$



$$\iint_D (xy+2x+3y)\,dA$$

$$= \int_0^1 \int_0^{1-y^2} (xy+2x+3y)\,dx\,dy$$

$$= \int_0^1 \left[y\frac{x^2}{2} + x^2 + 3xy \right]_{x=0}^{x=1-y^2} dy$$

$$= \frac{1}{2}\int_0^1 y(1-y^2)^2\,dy + \int_0^1 (1-y^2)^2\,dy + 3\int_0^1 y(1-y^2)\,dy$$

$u = 1-y^2$

$du = -2y\,dy$

$y=0 \to u=1$

$y=1 \to u=0$

$$= -\frac{1}{4}\int_1^0 u^2\,du - \frac{3}{2}\int_1^0 u\,du + \int_0^1 (1-2y^2+y^4)\,dy$$

$$= -\frac{1}{4}\,\frac{u^3}{3}\Big|_1^0 - \frac{3}{2}\,\frac{u^2}{2}\Big|_1^0 + \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1$$

$$= \frac{1}{12} + \frac{3}{4} + 1 - \frac{2}{3} + \frac{1}{5} = \boxed{\frac{41}{30}}$$

3. Sketch the region whose area is given by the integral $\int_0^\pi \int_1^{1+\sin\theta} r\,dr\,d\theta$.

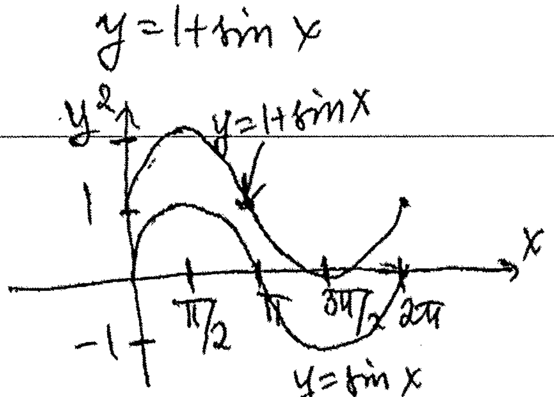
The region is bounded by $\theta=0$, $\theta=\pi$

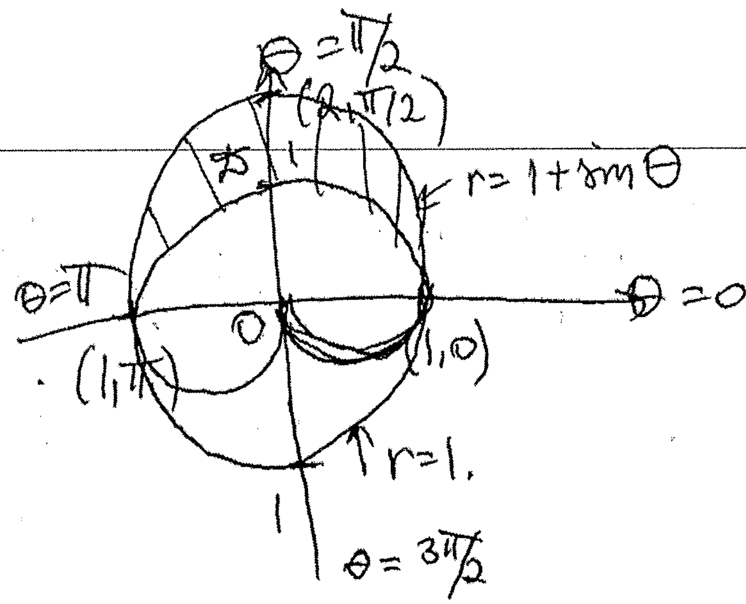
$r=1$

$r=1+\sin\theta$

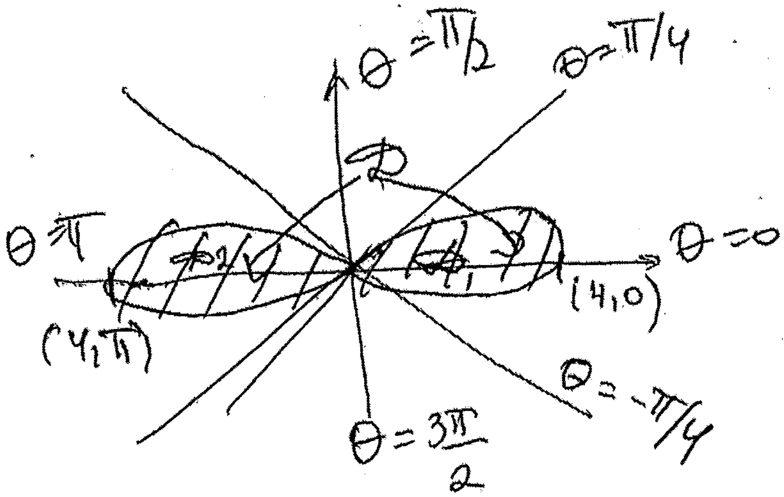
$r=1$ — circle of radius 1 centered at (

$r=1+\sin\theta$ -- cardioid.





4. Find the area of the region enclosed by the lemniscate $r^2 = 4\cos(2\theta)$.



$\mathcal{D}$ is the union of $\mathcal{D}_1$ and $\mathcal{D}_2$
since $A(\mathcal{D}_1) = A(\mathcal{D}_2)$
then $A(\mathcal{D}) = A = 2A(\mathcal{D}$

$$A = 2\iint_{\mathcal{D}_1} (1)\, dA = \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \\ 0 \le r \le \sqrt{4\cos 2\theta} \\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \end{array} \right|$$

$$= 2\int_{-\pi/4}^{\pi/4} \int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta = 2\int_{-\pi/4}^{\pi/4} \frac{r^2}{2} \Bigg|_{r=0}^{r=2\sqrt{\cos 2\theta}} d\theta$$

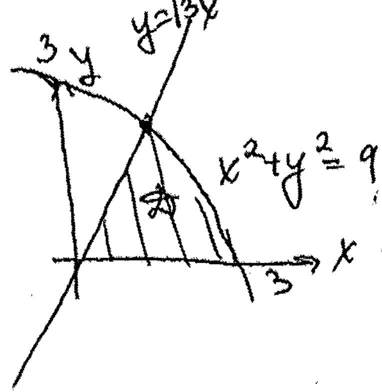
$$= \int_{-\pi/4}^{\pi/4} 4\cos^{\cancel{2}} 2\theta \, d\theta = \cancel{\int_{-\pi/4}^{\pi/4} 4\, \frac{1+\cos 4\theta}{2}\, d\theta}$$

$$\cancel{= 2\left(\theta + \frac{1}{4}\sin 4\theta\right)\Bigg|_{-\pi/4}^{\pi/4} = 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right)} =$$

$$= 4\,\frac{1}{2}\sin 2\theta \Bigg|_{-\pi/4}^{\pi/4} = 2\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = \boxed{4}$$

that lies in the first quadrant

5. Find the mass and center of mass of a lamina that occupies the region D bounded by the lines ~~$x=0$,~~ $y=0$, $y=\sqrt{3}x$ and the circle $x^2+y^2=9$ if the density function is $\rho(x,y)=xy^2$.



polar coordinates:

$x=r\cos\theta$

$y=r\sin\theta$

$dA=r\,dr\,d\theta$

$y=\sqrt{3}x$ in polar coon

$r\sin\theta=\sqrt{3}\,r\cos\theta$

$\tan\theta=\sqrt{3}$

$\theta=\pi/3$

$x^2+y^2=9$ in polar coord. $r=3$.

$0\le\theta\le\pi/3,\quad 0\le r\le 3$

$$m=\iint_D \rho(x,y)\,dA=\int_0^{\pi/3}\int_0^3 r\cos\theta\, r^2\sin^2\theta\, r\,dr\,d\theta=\int_0^{\pi/3}\cos\theta\sin^2\theta\,d\theta\cdot\int_0^3 r^4\,dr$$

$\sin\theta=u$

$du=\cos\theta\,d\theta$

$0\to 0$

$\pi/3\to\sqrt{3}/2$

$$=\int_0^{\sqrt{3}/2}u^2\,du\cdot\left.\frac{r^5}{5}\right|_0^3=\left.\frac{u^3}{3}\right|_0^{\sqrt{3}/2}\cdot\frac{(3)^5}{5}=\boxed{\frac{243\sqrt{3}}{40}}$$

$$\bar{x}=\frac{1}{m}\iint_D x\rho(x,y)\,dA=\frac{40}{243\sqrt{3}}\int_0^{\pi/3}\int_0^3 r^2\cos^2\theta\, r^2\sin^2\theta\, r\,dr\,d\theta$$

$$=\frac{40}{243\sqrt{3}}\int_0^{\pi/3}\cos^2\theta\sin^2\theta\,d\theta\int_0^3 r^5\,dr=\frac{40}{243\sqrt{3}}\int_0^{\pi/3}\frac{1}{4}\sin^2 2\theta\,d\theta\cdot\left.\frac{r^6}{6}\right|_0^3$$

$$=\frac{40}{243\sqrt{3}}\,\frac{1}{8}\int_0^{\pi/3}(1-\cos 4\theta)\,d\theta\cdot\frac{(3)^6}{6}=\frac{(5)(3)}{6\sqrt{3}}\left.\left(\theta-\frac{1}{4}\sin 4\theta\right)\right|_0^{\pi/3}$$

$$=\frac{5}{2\sqrt{3}}\left(\frac{\pi}{3}-\frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right)=\frac{5}{2\sqrt{3}}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right)$$

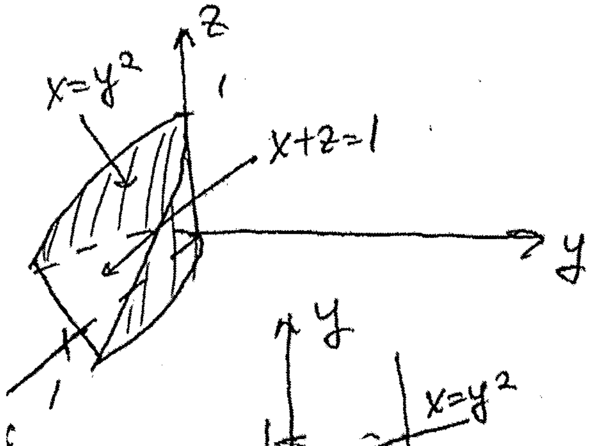
$$\bar{y}=\frac{1}{m}\iint_D y\rho(x,y)\,dA=\frac{40}{243\sqrt{3}}\int_0^{\pi/3}\int_0^3 (r\cos\theta)(r\sin\theta)^3\,r\,dr\,d\theta$$

$$=\frac{40}{243\sqrt{3}}\int_0^{\pi/3}\int_0^3 r^5\cos\theta\sin^3\theta\,dr\,d\theta=\frac{40}{243\sqrt{3}}\int_0^{\pi/3}\cos\theta\sin^3\theta\,d\theta\int_0^3 r^5\,dr$$

$$=\frac{40}{243\sqrt{3}}\int_0^{\sqrt{3}/2}u^3\,du\cdot\left.\frac{r^6}{6}\right|_0^3=\frac{40}{243\sqrt{3}}\cdot\frac{729}{6}\left.\frac{u^4}{4}\right|_0^{\sqrt{3}/2}=\frac{40}{18\sqrt{3}}\,\frac{1}{4}\left(\frac{9}{16}\right)=\frac{5}{16\sqrt{3}}$$

$$\boxed{(\bar{x},\bar{y})=\left(\frac{5}{2\sqrt{3}}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right),\ \frac{5}{16\sqrt{3}}\right)}$$

6. Evaluate $\iiint_E (x+2y)\,dV$ if E is bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.

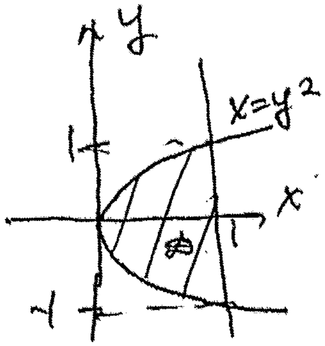


$0 \le z \le 1-x$

$-1 \le y \le 1$

$y^2 \le x \le 1$

$$\iiint_E (x+2y)\,dV = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} (x+2y)\,dz\,dx\,dy$$



$$= \int_{-1}^{1}\int_{y^2}^{1} (x+2y)z \Big|_{z=0}^{z=1-x} dx\,dy$$

$$= \int_{-1}^{1}\int_{y^2}^{1} (x+2y)(1-x)\,dx\,dy$$

$$\int_{-1}^{1}\int_{y^2}^{1} \left(x - x^2 + 2y - 2xy\right) dx\,dy = \int_{-1}^{1}\left[\frac{x^2}{2} - \frac{x^3}{3} + 2xy - x^2 y\right]\Big|_{x=y^2}^{x=1} dy$$

$$\int_{-1}^{1}\left[\frac{1}{2} - \frac{1}{3} + 2y - y - \frac{1}{2}(y^2)^2 + \frac{1}{3}(y^2)^3 + 2(y^2)y - (y^2)^2 y\right] dy$$

$$\int_{-1}^{1}\left(\frac{1}{6} + y - \frac{1}{2}y^4 + \frac{1}{3}y^6 + 2y^3 - y^5\right] dy$$

$$\left[\frac{1}{6}y + \frac{y^2}{2} - \frac{1}{2}\frac{y^5}{5} + \frac{1}{3}\frac{y^7}{7} + 2\frac{y^4}{4} - \frac{y^6}{6}\right]\Big|_{-1}^{1}$$

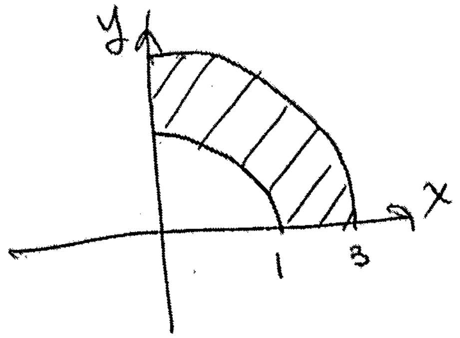
$$= \frac{1}{6}(2) - \frac{1}{10}(2) + \frac{1}{21}(2) = \frac{1}{3} - \frac{1}{5} + \frac{2}{21} = \boxed{\frac{8}{35}}$$

7. Sketch the solid whose volume is given by the integral $\int_1^3 \int_0^{\pi/2} \int_r^3 r\,dz\,d\theta\,dr$

The solid is bounded by
$r=1$, $r=3$ — cylinders in $\mathbb{R}^3$.
$\theta=0$, $\theta=\pi/2$
$r=z$ and $z=3$.
$z=3$ – plane parallel to the (xy)-plane

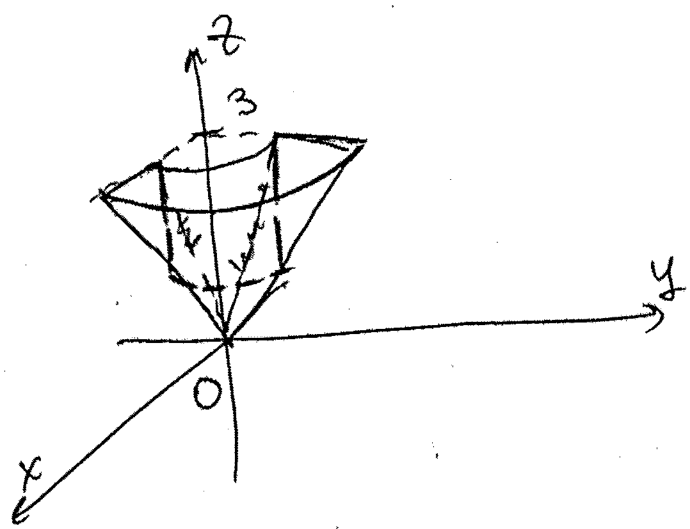
$z=r=\sqrt{x^2+y^2}$
$z^2=x^2+y^2$ – cone.

projection of the solid on the (xy)-plane is bounded by $r=1$ – circle of radius 1 centered at $(0,0)$

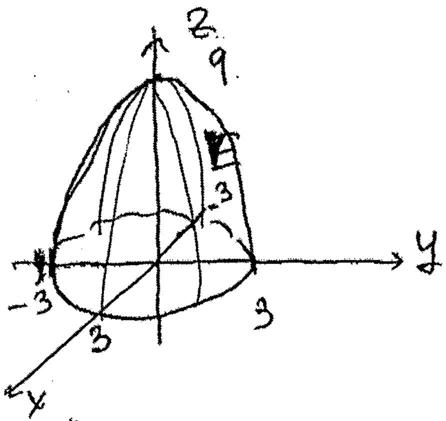


$r=3$ – circle of radius 3 centered at $(0,0)$

$\theta=0$, $\theta=\pi/2$.



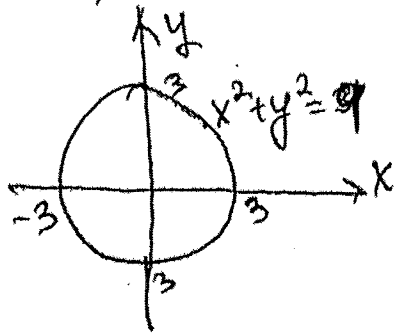
8. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.



cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

then the equation of the paraboloid $z = 9 - x^2 - y^2$ in cylindrical coordinates is



$$z = 9 - r^2$$

$$0 \le z \le 9 - r^2$$

$$0 \le r \le 3$$

$$0 \le \theta \le 2\pi.$$

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} \sqrt{r^2}\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 r^2 \, z\Big|_{z=0}^{z=9-r^2} dr$$

$$= 2\pi \int_0^3 r^2(9-r^2)\,dr$$

$$= 2\pi \int_0^3 (9r^2 - r^4)\,dr$$

$$= 2\pi\left(\frac{9r^3}{3} - \frac{r^5}{5}\right)\Big|_0^3 = 2\pi\left(3(3)^3 - \frac{(3)^5}{5}\right)$$

$$= 2\pi\left(81 - \frac{243}{5}\right) = 2\pi \cdot \frac{162}{5} = \boxed{\frac{324\pi}{5}}$$

9. Sketch the solid whose volume is given by the integral $\int_0^{2\pi} \int_0^{\pi/6} \int_1^3 \rho^2 \sin\varphi d\rho d\varphi d\theta$

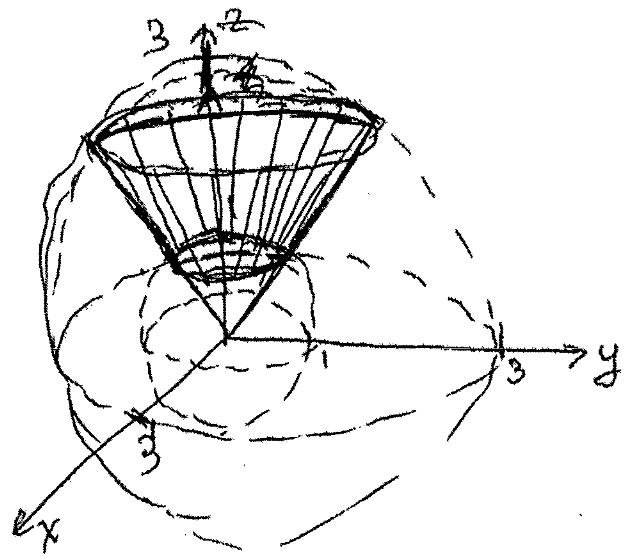
The solid is bounded by

$\theta = 0$, $\theta = 2\pi$

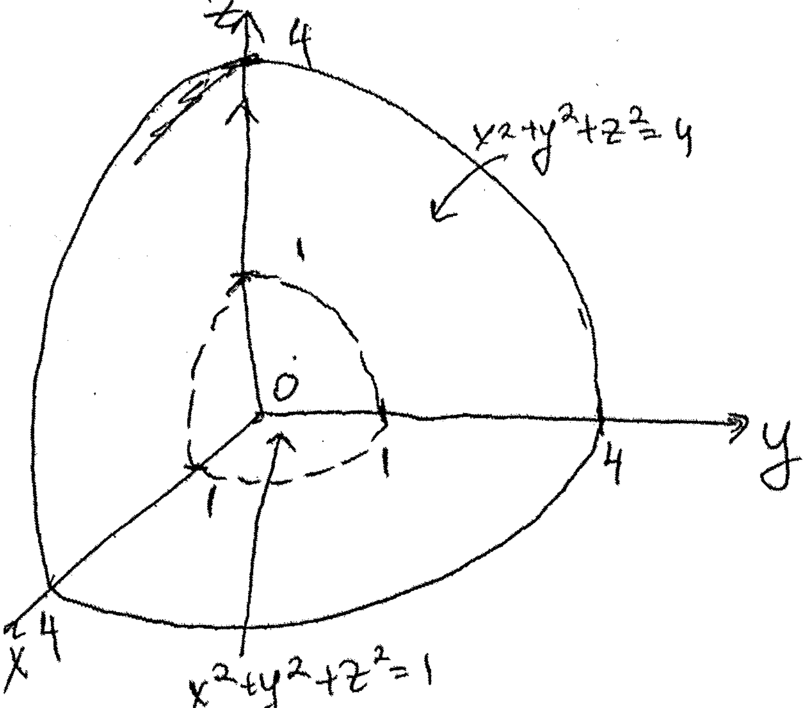
$\varphi = 0$, $\varphi = \pi/6$ – cone

$\rho = 1$ – sphere of radius 1 centered at (0,0

$\rho = 3$ – sphere of radius 3 centered at (0,0,0)



10. Evaluate $\iiint_E x e^{(x^2+y^2+z^2)^2}\,dV$ if the E is the solid that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.



Spherical coordinates:

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$

$x^2+y^2+z^2=\rho^2$

$1 \le \rho \le 2$

$0 \le \varphi \le \pi/2$

$0 \le \theta \le \pi/2.$

$$\iiint_E x e^{(x^2+y^2+z^2)^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho\cos\theta\sin\varphi\, e^{\rho^4}\rho^2\sin\varphi\,d\rho\,d\varphi$$

$$= \int_0^{\pi/2}\cos\theta\,d\theta \int_0^{\pi/2}\sin^2\varphi\,d\varphi \int_1^2 \rho^3 e^{\rho^4}\,d\rho \leftarrow$$

$\rho^4 = u$

$du = 4\rho^3\,d\rho$

$\rho = 1 \longrightarrow u = 1$

$\rho = 2 \longrightarrow u = 2^4 = 16$

$$= \sin\theta\Big|_0^{\pi/2} \cdot \int_0^{\pi/2}\frac{1-\cos 2\varphi}{2}\,d\varphi \cdot \frac{1}{4}\int_1^{16} e^u\,du$$

$$= (1)\cdot\frac{1}{2}\left(\varphi - \frac{1}{2}\sin 2\varphi\right)\Big|_0^{\pi/2} \cdot \frac{1}{4}e^u\Big|_1^{16}$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{1}{4}\left(e^{16}-e\right) = \boxed{\frac{\pi}{16}\left(e^{16}-e\right)}$$

11. Find the gradient vector field of the function $f(x, y, z) = xy^2 - yz^3$.

$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$

$f_x = y^2$

$f_y = 2xy - z^3$

$f_z = -3yz^2$

$\nabla f = \langle y^2, 2xy - z^3, -3yz^2 \rangle$

12. Evaluate the line integral $\int_C x^3 z\,ds$ if C is given by $x = 2\sin t$, $y = t$, $z = 2\cos t$, $0 \le t \le \pi/2$.

$$\int_C x^3 z\,ds = \int_0^{\pi/2} [x(t)]^3\,[z(t)]\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

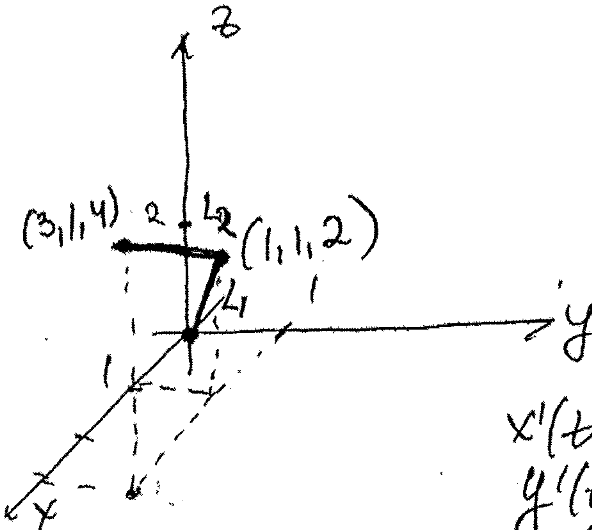
$$= \int_0^{\pi/2} 8\sin^3 t\; 2\cos t\sqrt{4\cos^2 t + 1 + 4\sin^2 t}\,dt$$

$$= \int_0^{\pi/2} 16\sin^3 t\cos t\sqrt{5}\,dt$$

$$= 16\sqrt{5}\int_0^{\pi/2}\sin^3 t\cos t\,dt = \left|\begin{array}{l} \sin t = u \\ du = \cos t\,dt \\ t=0 \longrightarrow u = \sin 0 = 0 \\ t = \pi/2 \longrightarrow u = \sin \pi/2 = 1 \end{array}\right.$$

$$= 16\sqrt{5}\int_0^1 u^3\,du = 16\sqrt{5}\,\frac{u^4}{4}\Big|_0^1 = \boxed{4\sqrt{5}}$$

13. Evaluate $\int_C y\,dx + z\,dy + x\,dz$ if C consists of the line segments from (0,0,0) to (1,1,2) and from (1,1,2) to (3,1,4).



L_1: $(0,0,0) \to (1,1,2)$

the line is parallel to $\vec{v} = \langle 1,1,2 \rangle$

a vector equation of L_1:

$\langle x,y,z \rangle = \langle 0,0,0 \rangle + t\langle 1,1,2 \rangle$

$$x'(t)=1 \quad y'(t)=1 \quad z'(t)=2 \qquad \begin{cases} x=t \\ y=t \\ z=2t \end{cases} \qquad 0 \le t \le 1$$

L_2: $(1,1,2) \to (3,1,4)$

the line is parallel to $\vec{v} = \langle 2,0,2 \rangle$

a vector equation of L_2:

$\langle x,y,z \rangle = \langle 1,1,2 \rangle + t\langle 2,0,2 \rangle$

$$x'(t)=2 \quad y'(t)=0 \quad z'(t)=2 \qquad \begin{cases} x=1+2t \\ y=1+0 \\ z=2+2t \end{cases} \qquad 0 \le t \le 1$$

$$\int_C y\,dx + z\,dy + x\,dz = \int_{L_1} y\,dx + z\,dy + x\,dz + \int_{L_2} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 [(t)(1) + (2t)(1) + (t)(2)]\,dt + \int_0^1 [(1)(2) + (2+2t)(0) + (1+2t)(2)$$

$$= \int_0^1 5t\,dt + \int_0^1 (4+4t)\,dt = \frac{5t^2}{2}\Big|_0^1 + \left(4t + \frac{4t^2}{2}\right)\Big|_0^1 = \frac{5}{2} + 4 + 2$$

$$= \frac{5}{2} + 6$$

$$= \boxed{\frac{17}{2}}$$

14. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = x^2y\vec{\imath} + e^y\vec{\jmath}$ and C is given by $\vec{r}(t) = t^2\vec{\imath} - t^3\vec{\jmath}$, $0 \le t \le 1$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy \quad \text{where}$$

$$P(x,y) = x^2y$$

$$Q(x,y) = e^y$$

$$x(t) = t^2 \qquad dx = 2t\,dt$$

$$y(t) = -t^3 \qquad dy = -3t^2\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left[(t^2)^2(-t^3)(2t) + e^{-t^3}(-3t^2)\right]dt$$

$$= \int_0^1 (-2t^8)\,dt + \int_0^1 (-3t^2)e^{-t^3}\,dt$$

$$\left|\begin{array}{l} u = -t^3 \\ du = -3t^2\,dt \\ t = 0 \rightarrow u = 0 \\ t = 1 \rightarrow u = -1 \end{array}\right|$$

$$= -2\frac{t^9}{9}\Big|_0^1 + \int_0^{-1} e^u\,du$$

$$= -\frac{2}{9} + e^u\Big|_0^{-1} = -\frac{2}{9} + (e^{-1} - e^0) = \cancel{-\frac{2}{9}(e^{-1}-e)}$$

$$\cancel{= \frac{2}{9}(1-e^{-1})}$$

15. Show that $\vec{F}(x,y) = (2x + y^2 + 3x^2y)\vec{i} + (2xy + x^3 + 3y^2)\vec{j}$ is conservative vector field. Use this fact to evaluate $\int_C \vec{F} \cdot d\vec{r}$ if C is the arc of the curve $y = x\sin x$ from $(0,0)$ to $(\pi, 0)$.

$P(x,y) = 2x + y^2 + 3x^2y \qquad Q(x,y) = 2xy + x^3 + 3y^2$

$\frac{\partial P}{\partial y} = 2y + 3x^2 \qquad \frac{\partial Q}{\partial x} = 2y + 3x^2$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\vec{F}$ is conservative.

f such that $\nabla f = \vec{F}(x,y)$:

$\nabla f = \langle f_x, f_y \rangle$

$$\begin{cases} f_x = 2x + y^2 + 3x^2y \\ f_y = 2xy + x^3 + 3y^2 \end{cases}$$

$f(x,y) = \int f_x \, dx = \int (2x + y^2 + 3x^2y)\,dx = x^2 + y^2x + x^3y + g(y)$

$g(y)$ is an unknown function.

plug f into the 2nd equation:

$\cancel{f_y = 2x + y^2 + 3x^2y + g}$ $\quad f_y = 2xy + x^3 + g'(y) = 2xy + x^3 + 3y^2$

$g'(y) = 3y^2$

$g(y) = y^3 + K$, K is a constant

$f(x,y) = x^2 + xy^2 + x^3y + y^3 + K$

$\int_C \vec{F} \cdot d\vec{r} = f(\pi, 0) - f(0,0) = \pi^2 + \pi \cdot 0 + \pi^3 \cdot 0 + 0^3 + K - K = \boxed{\pi^2}$

16. Show that $\vec{F}(x,y,z) = yz(2x+y)\vec{i} + xz(x+2y)\vec{j} + xy(x+y)\vec{k}$ is conservative vector field. Use this fact to evaluate $\int_C \vec{F}\cdot d\vec{r}$ if C is given by $\vec{r}(t) = (1+t)\vec{i} + (1+2t^2)\vec{j} + (1+3t^3)\vec{k}$, $0 \le t \le 1$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz+y^2z & x^2z+2xyz & x^2y+xy^2 \end{vmatrix}$$

$\vec{r}(0) = \cancel{\langle 0,0,0\rangle}\ \langle 1,1,1\rangle$

$\vec{r}(1) = \langle 2,3,4\rangle$.

$$= \vec{i}\left(\frac{\partial}{\partial y}(x^2y + xy^2) - \frac{\partial}{\partial z}(x^2z+2xyz)\right)$$

$$-\vec{j}\left(\frac{\partial}{\partial x}(x^2y+xy^2) - \frac{\partial}{\partial z}(2xyz+y^2z)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(x^2z+2xyz) - \frac{\partial}{\partial y}(2xyz+y^2z)\right)$$

$$= \vec{i}\,(x^2+2xy-(x^2+2xy)) - \vec{j}\,(2xy+y^2-(2xy+y^2)) + \vec{k}\,(2xz+2yz-(2xz)+2yz)$$

$= \vec{0} \rightarrow \vec{F}$ is conservative.

f such that $\nabla f = \langle f_x, f_y, f_z\rangle = \vec{F}$;

$$\begin{cases} f_x = 2xyz + y^2z \\ f_y = x^2z + 2xyz \\ f_z = x^2y + xy^2 \end{cases}$$

$$f = \int f_x\,dx = \int (2xyz + y^2z)\,dx = x^2yz + xy^2z + g(y,z)$$

$g(y,z)$ is unknown.

Plug f into the 2nd and the 3rd equations:

$$f_y = x^2z + 2xyz + g_y = x^2z + 2xyz$$

$g_y = 0$, thus g does not depend on y.

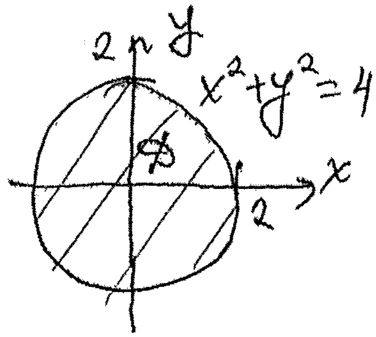
$$f_z = x^2y + xy^2 + g_z = x^2y + xy^2$$

$g_z = 0$, then $g(y,z) = K$ - constant.

$$f(x,y,z) = x^2yz + xy^2z + K.$$

$$\int_C \vec{F}\cdot d\vec{r} = f(2,3,4) - f(1,1,1) = (4)(3)(4) + (2)(9)(4) + K - 1 - 1 - K = \boxed{118}$$

17. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$ where C is the circle $x^2+y^2=4$ with counterclockwise orientation.



$$\int_C x^2y\,dx - x^4y^2\,dy = \iint_D \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right]$$

$$= \iint_D \left[-y^2 - x^2\right] dA = -\iint_D (x^2+y^2)\,dA =$$

polar coordinates!

$x = r\cos\theta \qquad dA = r\,dr\,d\theta$

$y = r\sin\theta \qquad x^2+y^2 = r^2$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

$$-\iint_D (x^2+y^2)\,dA = -\int_0^{2\pi}\int_0^2 r^2\, r\,dr\,d\theta = -\int_0^{2\pi} d\theta \int_0^2 r^3\,dr$$

$$= -2\pi \frac{r^4}{4}\Big|_0^2 = (-2\pi)(4)$$

$$= \boxed{-8\pi}$$

$$\begin{array}{ccc} P & Q & R \\ \| & \| & \| \end{array}$$

18. Find curl $\vec{F}$ and div $\vec{F}$ if $\vec{F} = x^2z\vec{\imath} + 2x\sin y\vec{\jmath} + 2z\cos y\vec{k}$.

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(2x\sin y) + \frac{\partial}{\partial z}(2z$$

$$= 2xz + 2x\cos y + 2\cos y$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & 2x\sin y & 2z\cos y \end{vmatrix}$$

$$= \vec{\imath}\left(\frac{\partial}{\partial y}(2z\cos y) - \frac{\partial}{\partial z}(2x\sin y)\right) - \vec{\jmath}\left(\frac{\partial}{\partial x}(2z\cos y) - \frac{\partial}{\partial z}(x^2z)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(2x\sin y) - \frac{\partial}{\partial y}(x^2z)\right)$$

$$= -2z\sin y\,\vec{\imath} + x^2\vec{\jmath} + 2\sin y\,\vec{k}$$

19. Show that there is no vector field $\vec{G}$ such that curl $\vec{G} = 2x\vec{\imath} + 3yz\vec{\jmath} - xz^2\vec{k}$.

Let us find

$$\operatorname{div}(\operatorname{curl}\vec{G}) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3yz) + \frac{\partial}{\partial z}(-xz^2)$$

$$= 2 + 3z - 2xz \neq 0$$

Since we know that for any vector field $\vec{F}$

$$\operatorname{div}(\operatorname{curl}\vec{F}) = 0$$

and $\operatorname{div}(\operatorname{curl}\vec{G}) \neq 0$, then there is no vector field $\vec{G}$ such that

$$\operatorname{curl}\vec{G} = \langle 2x, 3yz, -xz^2 \rangle.$$