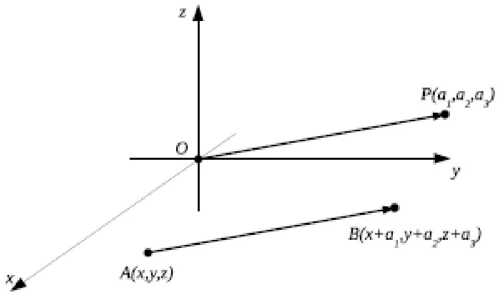
Chapter 11. **Three-dimensional analytic geometry and vectors**
Section 11.2 **Vectors and the dot product in three dimensions**

Geometrically, a three-dimensional vector can be considered as an arrow with both a length and direction. An arrow is a directed line segment with a starting point and an ending point. Algebraically, a **tree-dimensional vector** is an ordered triple $\vec{a} =< a_1, a_2, a_3 >$ of real numbers. The numbers a_1, a_2, and a_3 are called the **components** of $\vec{a}$.

A **representation** of the vector $\vec{a} =< a_1, a_2, a_3 >$ is a directed line segment $\vec{AB}$ from any point $A(x, y, z)$ to the point $B(x + a_1, y + a_2, z + a_3)$.

A particular representation of $\vec{a} =< a_1, a_2, a_3 >$ is the directed line segment $\vec{OP}$ from the origin to the point $P(a_1, a_2, a_3)$, and $\vec{a} =< a_1, a_2, a_3 >$ is called the **position vector** of the point $P(a_1, a_2, a_3)$.



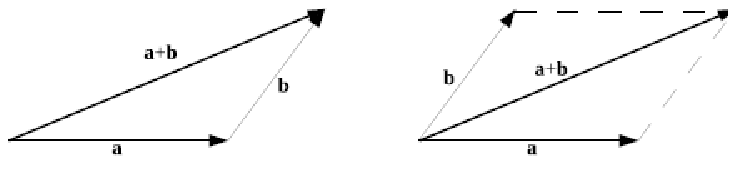
Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then $\boxed{\vec{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >}$.

The **magnitude (length)** $|\vec{a}|$ of $\vec{a}$ is the length of any its representation.

The length of $\vec{a} =< a_1, a_2, a_3 >$ is $\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$

The only vector with length 0 is the **zero vector** $\vec{0} =< 0, 0, 0 >$. This vector is the only vector with no specific direction.

Vector addition If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then the vector $\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$

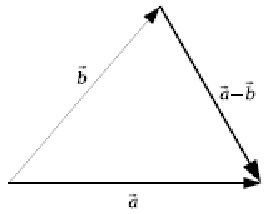


Triangle Law Parallelogram Law

Multiplication of a vector by a scalar If c is a scalar and $\vec{a} =< a_1, a_2, a_3 >$, then the vector $c\vec{a} =< ca_1, ca_2, ca_3 >$.

Two vectors $\vec{a}$ and $\vec{b}$ are called **parallel** if $\vec{b} = c\vec{a}$ for some scalar c. If $\vec{a} =< a_1, a_2, a_3 >$, $\vec{b} =< b_1, b_2, b_3 >$, then $\vec{a}$ and $\vec{b}$ are parallel if and only if $\boxed{\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}}$.

By the **difference** of two vectors, we mean $\boxed{\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >}$

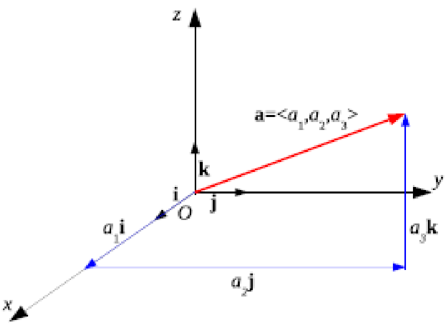


Properties of vectors If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k and m are scalars, then

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+\vec{0}=\vec{a}$
4. $\vec{a}+(-\vec{a})=\vec{0}$
5. $k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$
6. $(k+m)\vec{a}=k\vec{a}+m\vec{a}$
7. $(km)\vec{a}=k(m\vec{a})$
8. $1\vec{a}=\vec{a}$

Let $\vec{\imath}=<1,0,0>$ and $\vec{\jmath}=<0,1,0>$, $\vec{k}=<0,0,1>$, $|\vec{\imath}|=|\vec{\jmath}|=|\vec{k}|=1$.

$\boxed{\vec{a}=<a_1,a_2,a_3>=a_1\vec{\imath}+a_2\vec{\jmath}+a_3\vec{k}}$



A **unit vector** is a vector whose length is 1.

A vector $\boxed{\vec{u}=\frac{1}{|\vec{a}|}\vec{a}=\left\langle\frac{a_1}{|\vec{a}|},\frac{a_2}{|\vec{a}|},\frac{a_3}{|\vec{a}|}\right\rangle}$ is a unit vector that has the same direction as $\vec{a}=<a_1,a_2,a_3>$.

Example 1. Find the unit vector in the direction of the vector $\vec{\imath}-2\vec{\jmath}+2\vec{k}.=\vec{a}$

$$|\vec{a}|=\sqrt{1+4+4}=\sqrt{9}=3$$

$$\vec{u}=\frac{1}{3}<1,-2,2>$$

Definition. The **dot** or **scalar product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a}\cdot\vec{b} = 0$.

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then $\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$ and $\boxed{\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\pi/2$.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b} = 0$.

Example 2. Find the values of x such that the vectors $\vec{a} =< x, 1, 2 >$ and $\vec{b} =< 3, 4, x >$ are orthogonal.

$0 = \vec{a}\cdot\vec{b} = 3x + 4 + 2x$, then solve for x

$5x = -4$

$\boxed{x = -\frac{5}{4}}$

Direction angles and direction cosines. The **direction angles** of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\vec{a}$.

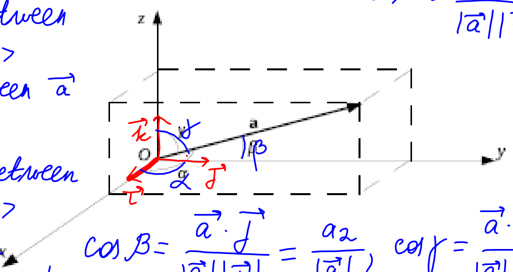
$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

α is the angle between $\vec{a}$ and $\vec{i} = <1, 0, 0>$

β is the angle between $\vec{a}$ and $\vec{j} = <0, 1, 0>$

γ is the angle between $\vec{a}$ and $\vec{k} = <0, 0, 1>$



$\cos\alpha = \dfrac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{\vec{a}\cdot\vec{j}}{|\vec{a}||\vec{j}|} = \dfrac{a_2}{|\vec{a}|}$, $\cos\gamma = \dfrac{\vec{a}\cdot\vec{k}}{|\vec{a}|\cdot|\vec{k}|} = \dfrac{a_3}{|\vec{a}|}$

$\boxed{\cos\alpha = \dfrac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \dfrac{a_1}{|\vec{a}|}}$, $\boxed{\cos\beta = \dfrac{a_2}{|\vec{a}|}}$, $\boxed{\cos\gamma = \dfrac{a_3}{|\vec{a}|}}$.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} = \frac{|\vec{a}|^2}{|\vec{a}|^2} = 1$$

We can write

$$\vec{a} =< a_1, a_2, a_3 >=< |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma >= |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$$

$\vec{a} = |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} =< \cos\alpha, \cos\beta, \cos\gamma >$$

which says that the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$.

Example 3. Find the direction cosines of the vector $< -4, -1, 2 >$. $= \vec{a}$

$$\cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{-4}{\sqrt{21}} \qquad |\vec{a}| = \sqrt{16+1+4} = \sqrt{21}$$

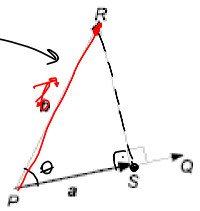
$$\cos\beta = \frac{a_2}{|\vec{a}|} = -\frac{1}{\sqrt{21}}$$

$$\cos\gamma = \frac{a_3}{|\vec{a}|} = \frac{2}{\sqrt{21}}$$

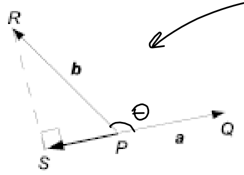
$0 < \theta < \frac{\pi}{2}$

ΔPRS:

$|\vec{PS}| = |\vec{b}|\cos\theta = \cancel{|\vec{b}|}\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cancel{|\vec{b}|}}$



$\frac{\pi}{2} < \theta < \pi$



$\vec{PS} = \text{proj}_{\vec{a}}\vec{b}$ is called the **vector projection of $\vec{b}$ onto $\vec{a}$**.

$|\vec{PS}| = \text{comp}_{\vec{a}}\vec{b}$ is called the **scalar projection of $\vec{b}$ onto** $\vec{a}$ or the **component of $\vec{b}$ along** $\vec{a}$. The scalar projection of $\vec{b}$ onto $\vec{a}$ is the length of the vector projection of $\vec{b}$ onto $\vec{a}$ if $0 \le \theta < \pi/2$ and is negative if $\pi/2 \le \theta < \pi$.

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}} \quad \boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} < a_1, a_2, a_3 >}$$

$$\vec{PS} = |\vec{PS}|\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\cdot\frac{\vec{a}}{|\vec{a}|}$$

Example 4. Find the scalar and vector projections of $\vec{b} = < 4, 2, 0 >$ onto $\vec{a} = < 1, 2, 3 >$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{4+4}{\sqrt{1+4+9}} = \boxed{\frac{8}{\sqrt{14}}}$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \boxed{\frac{8}{14}<1,2,3>}$$