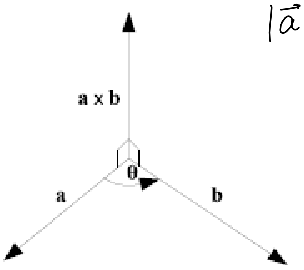
Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.3 **The cross product.**

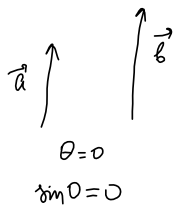
Definition. If $\vec{a}$ and $\vec{b}$ are nonzero three-dimensional vectors, the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

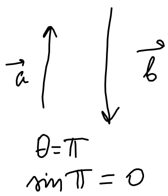
$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\vec{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and whose direction is given by the **right-hand rule:** If the fingers of your hand curl through the angle θ from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{n}$.

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$







If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, then we define $\vec{a} \times \vec{b}$ to be $\vec{0}$.

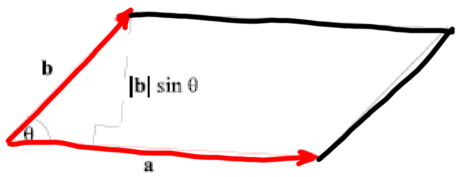
$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Properties of the cross product. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.



$A_{\text{parallelogram}} = |\vec{a} \times \vec{b}|$

The cross product in component form.

The cross product of a $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ is

$$\begin{array}{ccc} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$$= (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

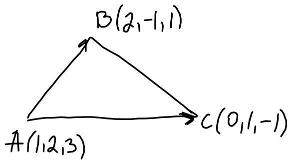
Example 1. If $\vec{a} = < -2, 3, 4 >$ and $\vec{b} = < 3, 0, 1 >$, find $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} = \vec{\imath}\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix}$$

$$= 3\vec{\imath} - \vec{\jmath}(-2-12) - 9\vec{k}$$

$$= \boxed{3\vec{\imath} + 14\vec{\jmath} - 9\vec{k}}$$

Example 2. Find the area of the triangle with vertices $A(1, 2, 3)$, $B(2, -1, 1)$, $C(0, 1, -1)$.



$$A_{\Delta} = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$$

$$\vec{AB} = \langle 1, -3, -2 \rangle$$
$$\vec{AC} = \langle -1, -1, -4 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & -2 \\ -1 & -1 & -4 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & -2 \\ -1 & -4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2 \\ -1 & -4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}$$

$$= \vec{i}(12-2) - \vec{j}(-4-2) + \vec{k}(-1-3)$$
$$= 10\vec{i} + 6\vec{j} - 4\vec{k}$$

$$A_{\Delta} = \frac{1}{2}\sqrt{100+36+16} = \boxed{\frac{\sqrt{152}}{2}}$$

Example 3. Find two unit vectors orthogonal to both $\vec{\imath}+\vec{\jmath}$ and $\vec{\imath}-\vec{\jmath}+\vec{k}$. $= \vec{b} = \langle 1,-1,1 \rangle$

$\langle 1,1,0 \rangle$

$\vec{a} = \langle 1,1,0 \rangle$

$\vec{c} = \vec{a} \times \vec{b} = \langle 1, -1, -2 \rangle$

$\vec{u}_1 = \frac{\vec{c}}{|\vec{c}|}$

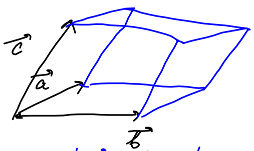
$\vec{u}_2 = -\frac{\vec{c}}{|\vec{c}|}$

Triple products

The product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the **scalar triple product** of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|.$$



Volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})|$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given in component form:

$$\vec{a} =< a_1, a_2, a_3 >, \quad \vec{b} =< b_1, b_2, b_3 >, \quad \vec{c} =< c_1, c_2, c_3 > .$$

Then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 4. Find the volume of the parallelepiped determined by vectors $\vec{a} = 2\vec{i}+3\vec{j}-2\vec{k}$, $\vec{b} = \vec{i}-\vec{j}$, and $\vec{c} = 2\vec{i}+3\vec{k}$.

$\vec{a} = < 2, 3, -2 >$

$\vec{b} = < 1, -1, 0 >$

$\vec{c} = < 2, 0, 3 >$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} \begin{matrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{matrix}$$

$= 2(-1)(3) + (3)(0)(2) + (-2)(1)(0)$

$\quad -\left(2(-1)(-2) + (0)(0)(2) + (3)(1)(3) \right)$

$= -6 - (4+9)$

$= -19$

$V = |-19| = \boxed{19}$

Vectors $\vec{a}, \vec{b}$, and $\vec{c}$ are complanar (lie in the same plane) if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

Example 5. Use the scalar triple product to verify that the vectors $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$, $\vec{b} = \vec{\imath} - \vec{\jmath}$, and $\vec{c} = 7\vec{\imath} + 3\vec{\imath} + 2\vec{k}$ are complanar; that is, they lie in the same plane.

The product $\vec{a} \times (\vec{b} \times \vec{c})$ is called the **vector triple product** of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.$$