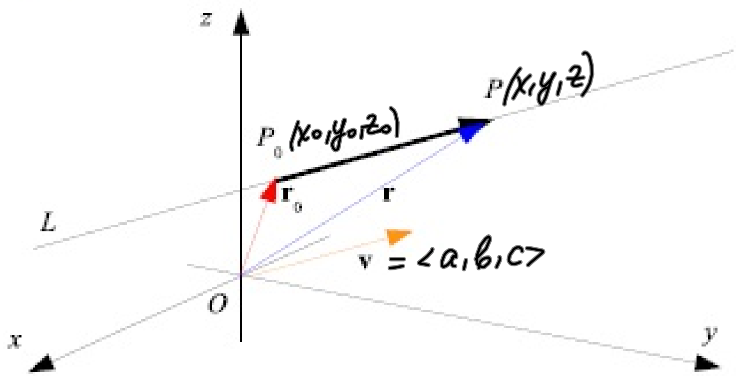
Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.4 **Equations of lines and planes.**

A line L in 3D space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. Let $\vec{v}$ be a vector parallel to L, $P(x, y, z)$ be an arbitrary point on L and $\vec{r_0}$ and $\vec{r}$ be position vectors of P_0 and P.



$\vec{r} = \vec{r_0} + \vec{P_0P}$. Since $\vec{P_0P}$ is parallel to $\vec{v}$, there is a scalar t such that $\vec{P_0P} = t\vec{v}$. Thus a **vector equation** of L is

$$\vec{r} = \vec{r_0} + t\vec{v}.$$

Each value of the **parameter** t gives the position vector $\vec{r}$ of a point on L.

If $\vec{r_0} =< x_0, y_0, z_0 >$, $\vec{r} =< x, y, z >$, and $\vec{v} =< a, b, c >$, then

vector equation : $<x,y,z> = <x_0,y_0,z_0> + t<a,b,c>$

$$< x, y, z >=< x_0 + ta, y_0 + tb, z_0 + tc >$$

or

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$ ← parametric equations

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} =< a, b, c >$.

If a vector $\vec{v} =< a, b, c >$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

Example 1. Find the vector equation and parametric equations for the line passing through the point $P(1, -1, -2)$ and parallel to the vector $\vec{v} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$.

x_0 y_0 z_0 $\qquad a \quad b \quad c$

$x_0 = 1 \qquad a = 3$

$y_0 = -1 \qquad b = -2$

$z_0 = -2 \qquad c = 1$

Vector equation: $\langle x, y, z \rangle = \langle 1, -1, -2 \rangle + t\langle 3, -2, 1 \rangle$

Parametric equations:

$$x = 1 + 3t$$
$$y = -1 - 2t$$
$$z = -2 + t$$

Since vectors $\vec{v} = < a, b, c >$ and $\vec{P_0P} = < x - x_0, y - y_0, z - z_0 >$ are parallel, then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

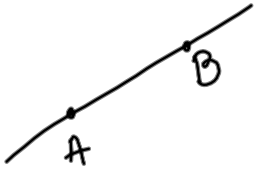
These equations are called **symmetric equations** of L. If one of a, b, or c is 0, we can still write symmetric equations. For instance, if $c = 0$, then the symmetric equations of L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0.$$

This means that L lies in the plane $z = z_0$.

Example 2. Find symmetric equations for the line passing through the given points:

(a) A $(2, -6, 1)$ B $(1, 0, -2)$

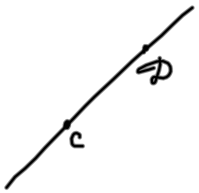


$\vec{AB}$ is parallel to the line

$\vec{AB} = \langle -1, 6, -3 \rangle$

$$\frac{x-2}{-1} = \frac{y+6}{6} = \frac{z-1}{-3} \quad \text{OR} \quad \frac{x-1}{-1} = \frac{y}{6} = \frac{z+2}{-3}$$

(b) C $(-1, 2, -4)$ D $(-1, -3, 2)$



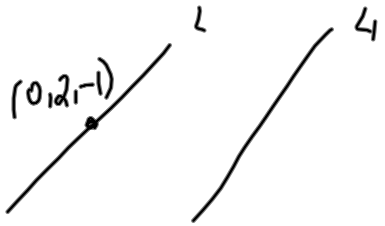
$\vec{CD} = \langle 0, -5, 6 \rangle$

$$x = -1 \qquad\qquad x = -1$$

$$\frac{y-2}{-5} = \frac{z+4}{6} \quad \text{OR} \quad \frac{y+3}{-5} = \frac{z-2}{6}$$

Line lies in the plane $x = -1$

Example 3. Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, and $z = 5 - 7t$.



$L \parallel L_1$

vector parallel to L_1 is $\langle 2, 3, -7 \rangle$

symmetric equations:

$$\frac{x-0}{2} = \frac{y-2}{3} = \frac{z+1}{-7}$$

Example 4. Determine whether the lines L_1 and L_2 are parallel, skew (do not intersect and are not parallel), or intersecting. If they intersect, find the point of intersection.

(a) $L_1 : \frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3}$, $L_2 : \frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{3}$

$L_1 \parallel \vec{v}_1 = \langle 2,4,-3\rangle$ $\quad L_2 \parallel \vec{v}_2 = \langle 1,3,3\rangle$

$\vec{v}_1 \nparallel \vec{v}_2$

$\boxed{\text{NOT PARALLEL}}$

POINT OF INTERSECTION.

1. Convert symmetric equations into parametric:

$\frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3} = t$

$$\begin{cases} x = 4+2t \\ y = -5+4t \\ z = 1-3t \end{cases}$$

$\frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{3} = s$

$$\begin{cases} x = 2+s \\ y = -1+3s \\ z = 3s \end{cases}$$

2. Set up a system for s and t.

$$\begin{array}{l} x: \\ y: \\ z: \end{array} \begin{cases} 4+2t = 2+s \\ -5+4t = -1+3s \\ 1-3t = 3s \end{cases}$$

3. Solve the system.

$$\begin{cases} 4+2t = 2+s \longrightarrow s = 2+2t \\ 1-3t = 3s \end{cases}$$

$3(2+2t) = 1-3t$

$6+6t = 1-3t$

$9t = -5$

$\boxed{t = -\frac{5}{9}} \longrightarrow s = 2+2\,\frac{-5}{9} = 2-\frac{10}{9} = \boxed{\frac{8}{9} = s}$

$-5+4\frac{(-5)}{9} \overset{?}{=} -1+3\frac{8}{9}$

$-5-\frac{20}{9} \qquad -1+\frac{24}{9}$

$-\frac{65}{9} \neq \frac{15}{9}$

$\boxed{\text{SKEW}}$

(b) $L_1: \dfrac{x-1}{2} = \dfrac{y}{1} = \dfrac{z-1}{4}$, $L_2: \dfrac{x}{1} = \dfrac{y+2}{2} = \dfrac{z+2}{3}$

$\vec{v}_1 = \langle 2,1,4 \rangle$, $\vec{v}_2 = \langle 1,2,3 \rangle$

$\vec{v}_1 \nparallel \vec{v}_2$

$\boxed{\text{NOT PARALLEL}}$

$$\frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{4} = t$$

$$\begin{cases} x = 1+2t \\ y = t \\ z = 1+4t \end{cases}$$

$$\frac{x}{1} = \frac{y+2}{2} = \frac{z+2}{3} = s$$

$$\begin{cases} x = s \\ y = -2+2s \\ z = -2+3s \end{cases}$$

$$\begin{array}{l} x: \\ y: \\ z: \end{array} \begin{cases} 1+2t = s \\ t = -2+2s \\ 1+4t = -2+3s \end{cases}$$

$$\begin{cases} 1+2t = s \\ t = -2+2s \end{cases}$$

$t = -2+2(1+2t)$

$t = -2+2+4t$

$\boxed{t=0} \longrightarrow s = 1+2(0) = \boxed{1 = s}$

$1+4(0) \overset{?}{=} -2+3(1)$

$1 = 1$

$\boxed{\text{INTERSECTING}}$

POINT OF INTERSECTION:

$x = 1+2(0)$
$y = 0$
$z = 1+4(0)$

OR

$x = 1$
$y = -2+2(1)$
$z = -2+3(1)$

$\boxed{(1,0,1)}$

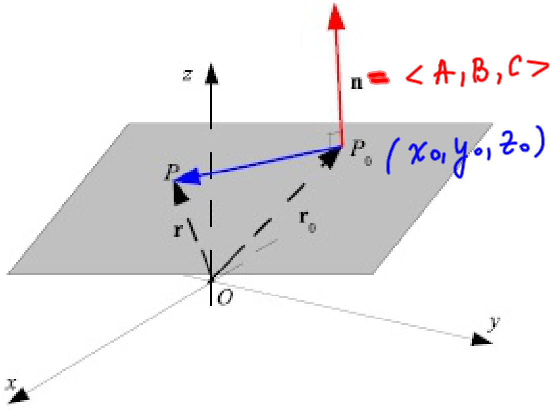
(c) $L_1: x = -6t, y = 1 + 9t, z = -3t,$ $\parallel \vec{v}_1 = < -6, 9, -3 >$

$L_2: x = 1 + 2s, y = 4 - 3s, z = s.$ $\parallel \vec{v}_2 = < 2, -3, 1 >$

$\vec{v}_1 \parallel \vec{v}_2$

$\boxed{\text{PA RALLEL}}$

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane. n is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane and $\vec{r}$ and $\vec{r_0}$ be the position vectors of P and P_0.



$\vec{P_0P} = \vec{r} - \vec{r_0}$. The normal vector $\vec{n}$ is orthogonal to every vector in the given plane, in particular, $\vec{n}$ is orthogonal to $\vec{P_0P}$.

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

Either of the two equations is called a **vector equation of the plane**.

Let $\vec{n} =< A, B, C >$, $\vec{r_0} =< x_0, y_0, z_0 >$, and $\vec{r} =< x, y, z >$, then

$$\vec{v} \cdot \vec{r} - \vec{r_0} = A(x - x_0) + B(y - y_0) + C(z - z_0)$$

so we can rewrite the vector equation in the following way:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

This equation is called the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\vec{n} =< A, B, C >$.

By collecting terms in a scalar equation of a plane, we can rewrite the equation as

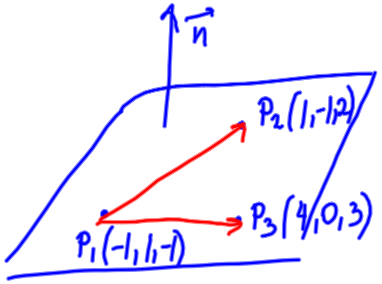
$$Ax + By + Cz + D = 0,$$

where $D = -Ax_0 - By_0 - Cz_0$.

Example 5. Find the equation of the plane through the point $P_0(\underset{x_0}{-5}, \overset{y_0}{1}, \underset{z_0}{2})$ with the normal vector $\vec{n} = \langle \underset{A}{3}, \underset{B}{-3}, \underset{C}{-1} \rangle$.

Equation: $\boxed{3(x+5) - 3(y-1) - 1(z-2) = 0}$

Example 6. Find the equation of the plane passing through the points $P_1(-1,1,-1)$, $P_2(1,-1,2)$, $P_3(4,0,3)$.



$\overrightarrow{P_1P_2} = \langle 2,-2,3\rangle$

$\overrightarrow{P_1P_3} = \langle 5,-1,4\rangle$

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 3 \\ 5 & -1 & 4 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -2 \\ 5 & -1 \end{vmatrix}$$

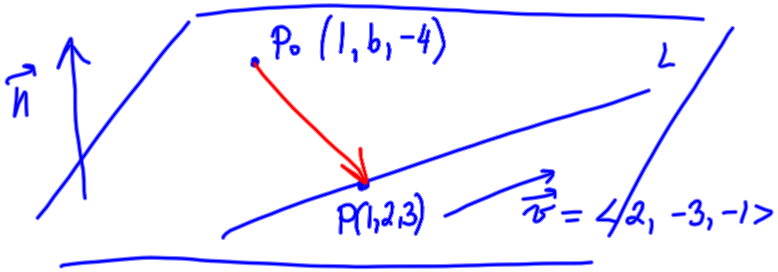
$$= \vec{i}(-8+3) - \vec{j}(8-15) + \vec{k}(-2+10)$$

$$= -5\vec{i} + 7\vec{j} + 8\vec{k}$$

$\boxed{-5(x+1)+7(y-1)+8(z+1)=0}$ OR $\boxed{-5(x-1)+7(y+1)+8(z-2)=0}$

OR $\boxed{-5(x-4)+7y+8(z-3)=0}$ OR $\boxed{-5x+7y+8z-4=0}$

Example 7. Find an equation of the plane that passes through the point $P_0(1, 6, -4)$ and contains the line $x = 1 + 2t$, $y = 2 - 3t$, $z = 3 - t$.



$$L: \begin{cases} x = 1+2t \\ y = 2-3t \\ z = 3-t \end{cases} \qquad L \parallel \vec{v} = \langle 2,-3,-1 \rangle$$

passes through $P(1,2,3)$

$\overrightarrow{P_0P}$ and $\vec{v}$

$$\overrightarrow{P_0P} = \langle 0,-4,7 \rangle$$

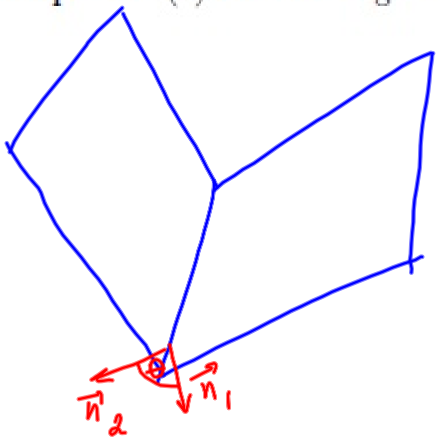
$$\vec{n} = \overrightarrow{P_0P} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -4 & 7 \\ 2 & -3 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} -4 & 7 \\ -3 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & 7 \\ 2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & -4 \\ 2 & -3 \end{vmatrix}$$

$$= 25\vec{i} + 14\vec{j} + 8\vec{k}$$

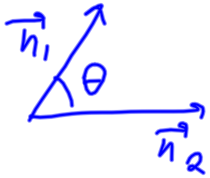
Equation: $\boxed{25(x-1)+14(y-6)+8(z+4)=0}$

Two planes are **parallel** if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

Example 8. (a) Find the angle between the planes $x - 2y + z = 1$ and $2x + y + z = 1$.



$\vec{n}_1 = \langle 1, -2, 1 \rangle$ $\quad \vec{n}_2 = \langle 2, 1, 1 \rangle$



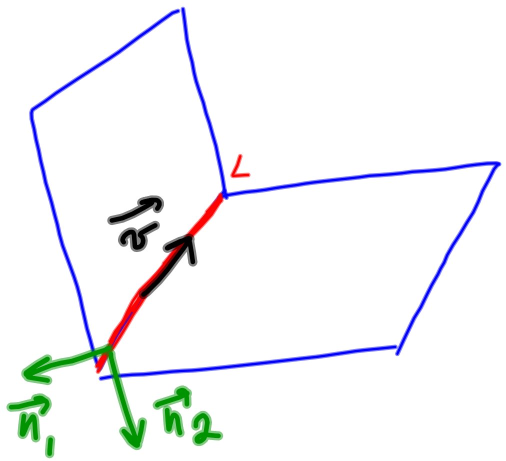
$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

$$= \frac{2-2+1}{\sqrt{1+4+1}\,\sqrt{1+4+1}} = \frac{1}{6}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{6}\right)}$$

(b) Find symmetric equation for the line of intersection of the planes.

$x - 2y + z = 1 \quad , \quad 2x + y + z = 1$



$\vec{v}$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$

$\vec{v} = \vec{n}_1 \times \vec{n}_2$

$\vec{n}_1 = \langle 1, -2, 1 \rangle$

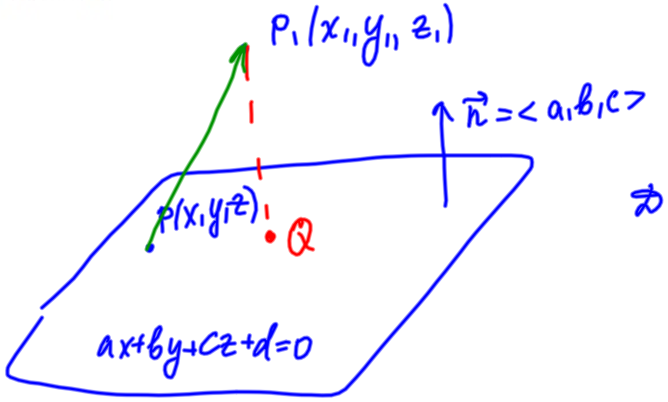
$\vec{n}_2 = \langle 2, 1, 1 \rangle$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$$= -3\vec{i} + \vec{j} + 5\vec{k}$$

$$\begin{cases} x - 2y + z = 1 \\ 2x + y + z = 1 \end{cases} \qquad (0, x, z$$

Problem. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.



$$\overrightarrow{PP_1} = \langle x_1 - x,\ y_1 - y,\ z_1 - z \rangle$$

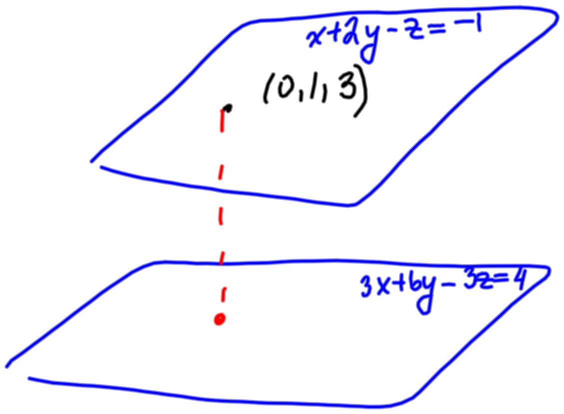
$$D = |\overrightarrow{PQ}| = \text{comp}_{\vec{n}}\,\overrightarrow{PP_1}$$

$$= \frac{\overrightarrow{PP_1} \cdot \vec{n}}{|\vec{n}|}$$

$$= \boxed{\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = D}$$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 9. Find the distance between the parallel planes $x+2y-z = -1$ and $3x+6y-3z = 4$.



$(0,1,3)$

$3x+6y-3z-4=0$

$$D = \frac{|3(0)+6(1)-3(3)-4|}{\sqrt{9+36+9}}$$

$$= \boxed{\frac{7}{\sqrt{54}}}$$