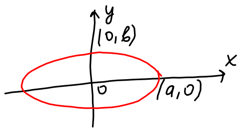
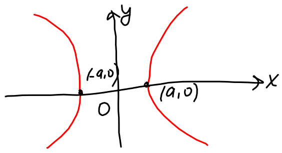
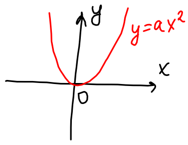
Section 11.5 **Quadric surfaces.**

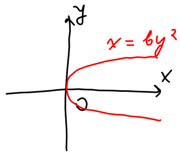
Curves in $\mathbb{R}^2$:

ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ parabola $y = ax^2$ or $x = by^2$









A **quadric surface** is the graph of a second degree equation in three variables. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, ..., J$ are constants. By translation and rotation the equation can be brought into one of two standard forms

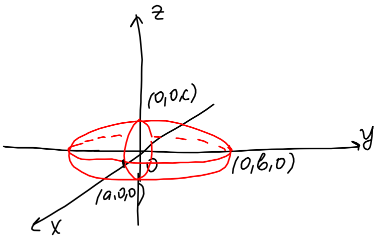
$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

In order to sketch the graph of a quadric surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces of the surface.**

Ellipsoids. The quadric surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an ellipsoid because all of its traces are ellipses.



six intersepts:

$(\pm a, 0, 0)$

$(0, \pm b, 0)$

$(0, 0, \pm c)$

all of points lie in the box

$|x| \le a$

$|y| \le b$

$|z| \le c$

Ellipsoid is symmetric with respect to coordinate planes.

The six intercepts of the ellipsoid are $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$ and the ellipsoid lies in the box $|x| \le a$, $|y| \le b$, $|z| \le c$

Since the ellipsoid involves only even powers of x, y, and z, the ellipsoid is symmetric with respect to each coordinate plane.

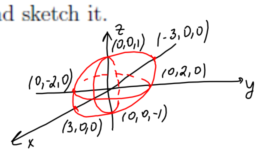
Example 1. Find the traces of the surface

$$4x^2 + 9y^2 + 36z^2 = 36$$

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

$$\frac{4x^2 + 9y^2 + 36z^2 = 36}{36}$$

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1 \quad \text{ellipsoid}$$



Traces

1) $x = k$, k is a constant $\quad \parallel (yz)$-plane

$$4k^2 + 9y^2 + 36z^2 = 36$$

$$\frac{9y^2 + 36z^2 = 36 - 4k^2}{36 - 4k^2}$$

$$\boxed{\frac{9}{36-4k^2}y^2 + \frac{36}{36-4k^2}z^2 = 1 \quad \text{ellipse}}$$

2) $y = k \quad \parallel (xz)$-plane

$$4x^2 + 9k^2 + 36z^2 = 36$$

$$\frac{4x^2 + 36z^2 = 36 - 9k^2}{36 - 9k^2}$$

$$\boxed{\frac{4}{36-9k^2}x^2 + \frac{36}{36-9k^2}z^2 = 1 \quad \text{ellipse}}$$

3) $z = k \quad \parallel (xy)$-plane

$$4x^2 + 9y^2 + 36k^2 = 36$$

$$\frac{4x^2 + 9y^2 = 36 - 36k^2}{36 - 36k^2}$$

$$\boxed{\frac{4}{36-36k^2}x^2 + \frac{9}{36-36k^2}y^2 = 1 \quad \text{ellipse}}$$

Hyperboloids.

- **Hyperboloid of one sheet.** The quadric surface with equations

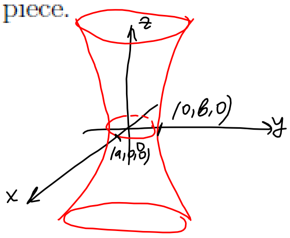
  1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
  2. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
  3. $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

  are called hyperbolids of one sheet since two of its traces are hyperbolas and one is an ellipse, and its surface consists of just one piece.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces in horizontal planes $z = k$ are ellipses

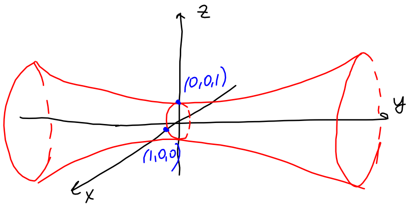
Traces in vertical planes are hyperbolas.



Example 2. Find the traces of the surface

$$x^2 - y^2 + z^2 = 1$$ hyperboloid of one sheet oriented along the y-axis

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.



Traces. $x^2-y^2+z^2=1$

1) $x=k$

$k^2-y^2+z^2=1$

$-y^2+z^2=1-k^2$

$\boxed{-\frac{y^2}{1-k^2}+\frac{z^2}{1-k^2}=1 \quad \text{hyperbola}}$

2) $y=k$

$x^2-k^2+z^2=1$

$x^2+z^2=1+k^2$

$\boxed{\frac{x^2}{1+k^2}+\frac{z^2}{1+k^2}=1 \quad \text{circle}}$

3) $z=k$

$x^2-y^2+k^2=1$

$x^2-y^2=1-k^2$

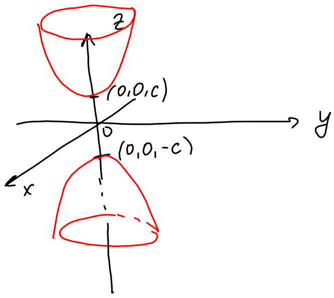
$\boxed{\frac{x^2}{1-k^2}-\frac{y^2}{1-k^2}=1 \quad \text{hyperbola}}$

- **Hyperboloid of two sheets.** The quadric surface with equations

1. $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

are called hyperbolids of two sheets since two of its traces are hyperbolas and one is an ellipse, and its surface is two sheets.

$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



Example 3. Find the traces of the surface

$$9x^2 - y^2 - z^2 = 9$$

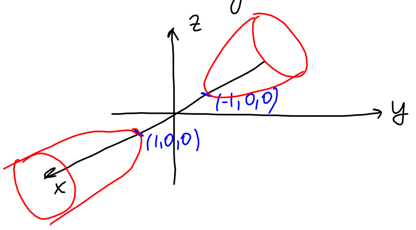
3

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

$$\frac{9x^2 - y^2 - z^2 = 9}{9}$$

$$x^2 - \frac{y^2}{9} - \frac{z^2}{9} = 1$$

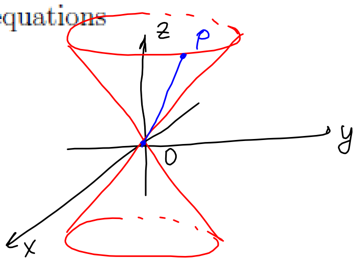
oriented along the x-axis.



Cone. The quadric surface with equations

1. $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$
2. $\dfrac{y^2}{b^2} = \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$
3. $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

are called cones.



horizontal traces are ellipses

vertical traces in $y=k$ and $x=k$ are hyperbolas if $k \neq 0$ and are pair of lines if $k=0$.

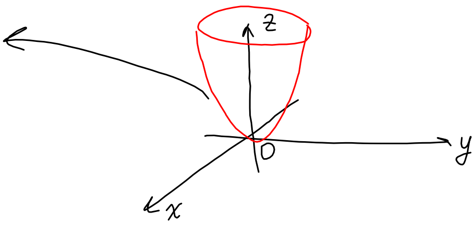
For any point P on the cone, the segment OP lies on the cone.

If P is any point on the cone, then the line OP lies entirely on the cone. The traces in horizontal planes $z = k$ are ellipses and traces in vertical planes $x = k$ or $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

Paraboloids.

- **Elliptic paraboloids.** The quadric surface with equations

1. $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$
2. $\dfrac{y}{b} = \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$
3. $\dfrac{x}{a} = \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$



vertical traces are parabolas

horizontal traces are ellipses

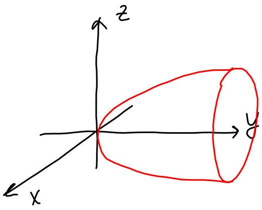
are called **elliptic paraboloids** because its traces in horizontal planes $z = k$ are ellipses, whereas its in vertical planes $x = k$ or $y = k$ are parabolas.

Example 4. Find the traces of the surface

$$y = x^2 + z^2$$

elliptic paraboloid oriented along the y-axis

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

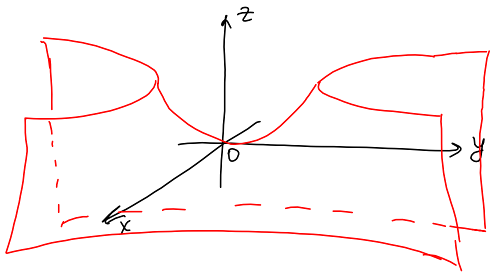


- **Hyperbolic paraboloids.** The quadric surface with equations

  1. $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$
  2. $\dfrac{y}{b} = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$
  3. $\dfrac{x}{a} = \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$

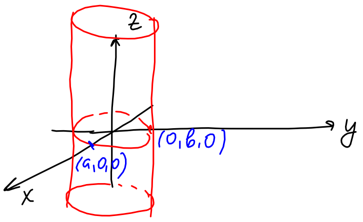
  are called **hyperbolic paraboloids** because its traces in horizontal planes $z = k$ are ~~ellipses~~ hyperbolas, whereas its in vertical planes $x = k$ or $y = k$ are ~~hyperbolas~~ parabolas.

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

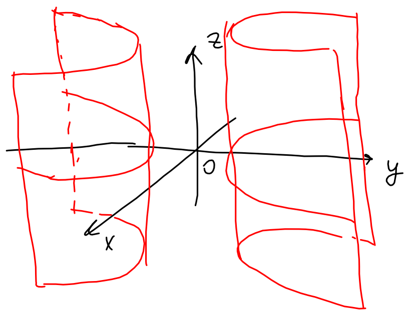


Quadric cylinders. When one of the veriables is missing from the equation of a surface, then a surface is a cylinder.

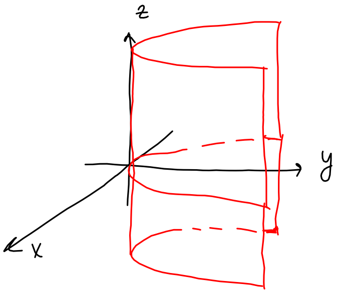
- The equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ represents the elliptic cylinder



- The equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ represents the hyperbolic cylinder



- The equation $y = ax^2$ represents the parabolic cylinder



Example 5. Classify the surface

$$4x^2 - y^2 + z^2 + 8x + 8z + 24 = 0$$

and sketch it.

Complete squares:

$$(4x^2+8x) - y^2 + (z^2+\overset{4\cdot 2}{8z}) + 24 = 0$$

$$4(x^2+2x+1) - 4\cdot 1 - y^2 + (z^2+8z+16) - 16 + 24 = 0$$

$$4(x+1)^2 - y^2 + (z+4)^2 + 4 = 0$$

$$\frac{4 = -4(x+1)^2 + y^2 - (z+4)^2}{4}$$

$$1 = -(x+1)^2 + \frac{y^2}{\underset{2^2}{4}} - \frac{(z+4)^2}{4}$$

$(-1, 0, -4)$

hyperboloid of two sheets

the axis of the hyperboloid is parallel to the y-axis

$|y| < 2$ – hyperboloid is not defined