Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.6 **Vector functions and space curves.**

Let $\vec{r}$ be a **vector function** whose range is a set of three-dimensional vectors.

$$\vec{r}(t) =< f(t), g(t), h(t) >= f(t)\vec{\imath} + g(t)\vec{\jmath} + h(t)\vec{k}$$

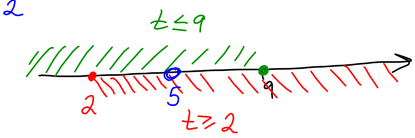
Functions f, g, and h are real-valued functions called the **component functions** of $\vec{r}$.
The domain of $\vec{r}$ consists of all values of t for which the expression for $\vec{r}(t)$ is defined.

Example 1. Find the domain of the vector function $\vec{r}(t) = \left\langle \sqrt{9-t}, \sqrt{t-2}, \dfrac{e^t}{t-5} \right\rangle$.

$f(t) = \sqrt{9-t}$ $\quad 9-t \geq 0 \quad t \leq 9$

$g(t) = \sqrt{t-2}$ $\quad t-2 \geq 0 \quad t \geq 2$

$h(t) = \dfrac{e^t}{t-5}$ $\quad t \neq 5$



$\boxed{\text{Domain: } [2,5) \cup (5,9]}$

Definition. If $\vec{r}(t) =< f(t), g(t), h(t) >$, then

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

provided the limits of the component function exist.

Example 2. Find the limit

$$\lim_{t\to 0} \left\langle \frac{1-\cos t}{t}, t^3, e^{-1/t^2} \right\rangle.$$

$$= < \lim_{t\to 0} \frac{1-\cos t}{t}, \cancel{\lim_{t\to 0} t^3}^{\,0}, \cancel{\lim_{t\to 0} e^{-1/t^2}}^{\,0} >$$

$$= < \lim_{t\to 0} \frac{\sin t}{1}, 0, 0 >$$

$$= \boxed{< 0, 0, 0 >}$$

Definition. A vector function $\vec{r}$ is **continuous at** a if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.
$\vec{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t), \quad y = g(t) \quad z = h(t)$$

ant t varies throughtout the interval I, is called a **space curve**. Equations

$$x = f(t), \quad y = g(t) \quad z = h(t)$$

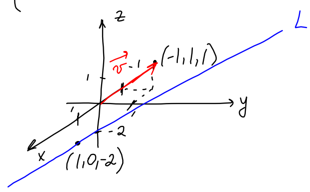
are called **parametric equations of C** and t is called a **parameter**.

Example 3. Sketch the curve with the given vector equation.

1. $\vec{r}(t) =< 1 - t, t, t - 2 >$

$$\begin{cases} x = 1 - t \\ y = t \\ z = t - 2 \end{cases}$$

line L passes through $(1, 0, -2)$ and is parallel to the vector $(-1, 1, 1)$

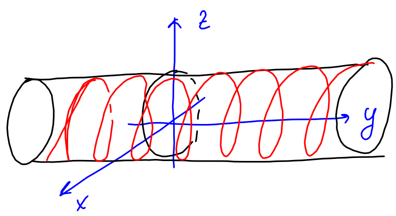


2. $\vec{r}(t) =< \cos 4t, t, \sin 4t >$

Parametric equations:

$$\begin{cases} x = \cos 4t \\ y = t \\ z = \sin 4t \end{cases}$$

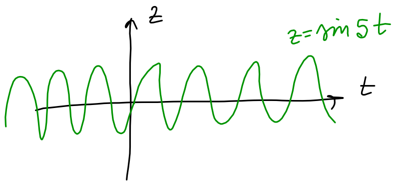
$x^2 + z^2 = 1$ circular cylinder directed along the y-axis



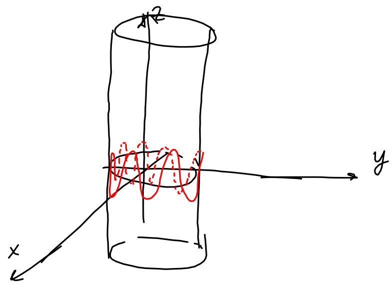
3. $\vec{r}(t) =< \cos t, \sin t, \sin 5t >$

$x(t) = \cos t$
$y(t) = \sin t$
$z(t) = \sin 5t$

$x^2 + y^2 = 1$

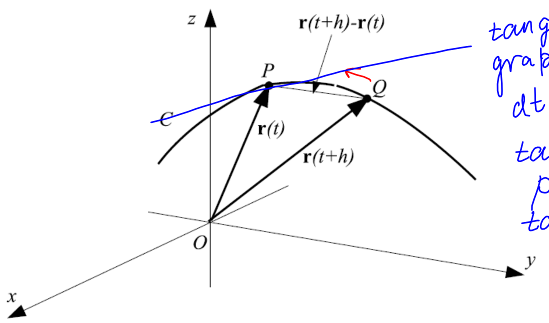


$\boxed{-1 \le z \le 1}$



Derivatives and integrals. The **derivative** $\vec{r}'$ of a vector function $\vec{r}$ is

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$



tangent line to the graph of $\vec{r}(t)$ at the point P

tangent line is parallel to the tangent vector $\vec{r}\,'(t)$.

The vector $\vec{r}\,'(t)$ is called the **tangent vector** to the curve defined by $\vec{r}$ at the point P, provided that $\vec{r}\,'(t)$ exists and $\vec{r}\,'(t) \neq \vec{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\vec{r}\,'(t)$. The **unit tangent vector**

$$\boxed{\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}}$$

Theorem. If $\vec{r}(t) =< f(t), g(t), h(t) >$, then

$$\boxed{\vec{r}\,'(t) =< f'(t), g'(t), h'(t) >}$$

Example 4. Find the derivative of the vector function $\vec{r}(t) = \ln(4-t^2)\vec{\imath} + \sqrt{1+t}\vec{\jmath} - 4e^{3t}\vec{k}$.

$= < \ln(4-t^2), \sqrt{1+t}, -4e^{3t} >$

$$\vec{r}\,'(t) = < \frac{-2t}{4-t^2}, \frac{1}{2}(1+t)^{-1/2}, -12e^{3t} >$$

Example 5. At what point do the curves $\vec{r}_1(t) =< t, 1-t, 3+t^2 >$ and $\vec{r}_2(s) =< 3-s, s-2, s^2 >$ intersect? Find their angle of intersection.

$\vec{r}_1(t) = < t, 1-t, 3+t^2 >$, $\vec{r}_2(s) = < 3-s, s-2, s^2 >$

1) Point of intersection.

Parametric equations:

$\vec{r}_1(t): \begin{cases} x = t \\ y = 1-t \\ z = 3+t^2 \end{cases}$ $\qquad \vec{r}_2(s): \begin{cases} x = 3-s \\ y = s-2 \\ z = s^2 \end{cases}$

$\begin{cases} t = 3-s \\ 1-t = s-2 \\ 3+t^2 = s^2 \end{cases}$ solve for s and t

$\cancel{\begin{cases} t = 3-s \\ 1-t = s-2 \end{cases}} \longrightarrow \cancel{1-(3-s) = s-2}$

$\cancel{1-3+s = s-2}$

$\cancel{s-2 = s-2}$

$\begin{cases} t = 3-s \\ 3+t^2 = s^2 \end{cases}$

$3+(3-s)^2 = s^2$

$3+9-6s+\cancel{s^2} = \cancel{s^2}$

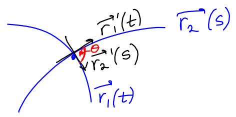
$12-6s = 0$

$\boxed{s=2}$ $\boxed{t=1}$

Point of intersection $\vec{r}_1(1) = (1,0,4)$

$\vec{r}_2(2) = \boxed{(1,0,4)}$

2)



Tangent vectors:

$\vec{r}_1\,'(t) = < 1, -1, 2t >$

$\vec{r}_2\,'(s) = < -1, 1, 2s >$

Tangent vectors at $(1,0,4)$

$\vec{r}_1\,'(1) = < 1, -1, 2 >$

$\vec{r}_2\,'(2) = < -1, 1, 4 >$

$$\cos\theta = \frac{< 1,-1,2 > \cdot < -1,1,4 >}{\sqrt{1+1+4}\,\sqrt{1+1+16}} = \frac{6}{\sqrt{6}\sqrt{18}} = \frac{1}{\sqrt{3}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

Theorem. Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\vec{u}(t)+\vec{v}(t)] = \vec{u}'(t)+\vec{v}'(t)$

2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$

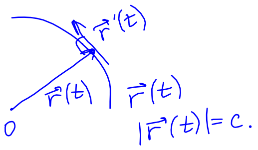
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t)+f(t)\vec{u}'(t)$

4. $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \vec{u}'(t)\cdot\vec{v}(t)+\vec{u}(t)\cdot\vec{v}'(t)$

5. $\frac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] = \vec{u}'(t)\times\vec{v}(t)+\vec{u}(t)\times\vec{v}'(t)$

6. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(t)$

Example 6. Show that if $|\vec{r}(t)| = c$, where c is a constant, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t.



$\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|\cdot|\vec{r}(t)|\cos 0$
$= |\vec{r}(t)|^2 = c^2$, c is a constant

$\vec{r}(t)\cdot\vec{r}(t) = c^2$

$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \frac{d}{dt}(c^2)$

$\vec{r}'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}'(t) = 0$

$2\vec{r}(t)\cdot\vec{r}'(t) = 0$

OR $\vec{r}(t)\cdot\vec{r}'(t) = 0 \longrightarrow \boxed{\vec{r}(t)\perp\vec{r}'(t)}$

The **definite integral** of a continuous vector function $\int_a^b \vec{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$

The Fundamental Theorem of Calculus for continuous vector functions says that

$$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big]_a^b = \vec{R}(b)-\vec{R}(a)$$

where $\vec{R}$ is an antiderivative of $\vec{r}$. We use the notation $\int \vec{r}(t)dt$ for indefinite integrals (antiderivatives).

Example 7. Find $\vec{r}(t)$ if $\vec{r}'(t) =< \sin t, -\cos t, 2t >$ and $\vec{r}(0) = \vec{\imath}+\vec{\jmath}+2\vec{k}$.
$=<1,1,2>$

$\vec{r}(t) = \int \vec{r}'(t)dt = < \int \sin t\,dt, \int(-\cos t)dt, \int 2t\,dt >$
$= < -\cos t + C_1, -\sin t + C_2, t^2 + C_3 >$

$<1,1,2> = \vec{r}(0) = < -1+C_1, C_2, C_3 >$

$\begin{cases} 1 = -1+C_1 \\ 1 = C_2 \\ 2 = C_3 \end{cases}$ $\quad C_1 = 2$, $C_2 = 1$, $C_3 = 2$

$\boxed{\vec{r}(t) = < -\cos t + 2, -\sin t + 1, t^2 + 2 >}$