Chapter 12. **Partial derivatives.**
Section 12.1 **Functions of several variables.**

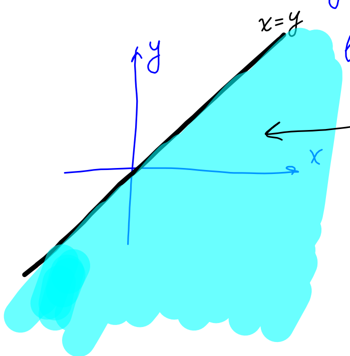
Definition. Let $D \subset \mathbb{R}^2$. A **function** f **of two variables** is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y)|(x, y) \in D)\}$.

We write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is **dependent variable**.

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is well-defined real number.

Example 1. Find the domain and the range of the function $f(x, y) = x^2 \ln(x - y)$ and evaluate $f(e, 0)$.

$f(x,y) = x^2 \ln(x-y)$



$\ln(x-y)$ is defined if $x-y>0$

Domain: set of (x,y) such that $x-y>0$.

$\boxed{\text{Range: } (-\infty, \infty)}$

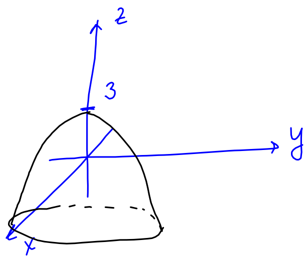
$f(e,0) = (e^2) \ln(e-0)$
$= e^2 \ln e$
$= \boxed{e^2}$

Definition. If f is a function of two variables with domain D, the **graph** of f is the set

$$S = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y), (x, y) \in D\}.$$

Example 2. Sketch the graph of the function $f(x, y) = 3 - x^2 - y^2$.

$z = 3 - x^2 - y^2$ — 1) circular paraboloid
2) oriented along negative z-axis
3) vertex $(0,0,3)$



level curves = traces of a surface in plane $z=k$.

Definition. The **level curves** of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant (in the range of f).

A level curve $f(x,y)=k$ is the locus of all points at which f takes on a given value k. In other words, it shows where the graph of f has height k.

Example 3. Describe the level curves for the following functions.

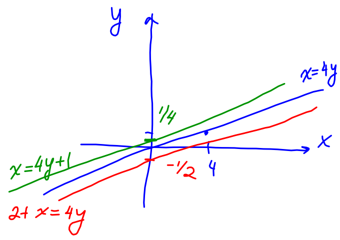
1. $f(x,y)=-x+4y$ lines parallel to the line $x=4y$

level curves $k=-x+4y$

$k=0$ $-x+4y=0$ $x=4y$

$k=1$ $-x+4y=1$

$k=-2$ $-x+4y=-2$

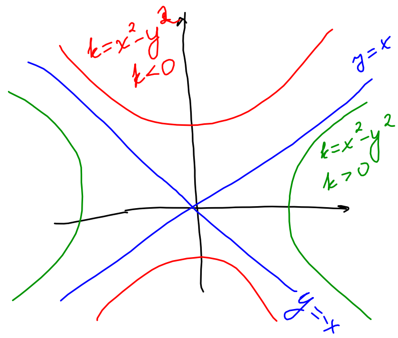


2. $f(x,y)=x^2-y^2$

level curves $k=x^2-y^2$

$k\neq 0$ — hyperbolas

$k=0$ – pair of lines $x=\pm y$



Functions of three or more variables.

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

We can get some information about f by examining its **level surfaces**, which are surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

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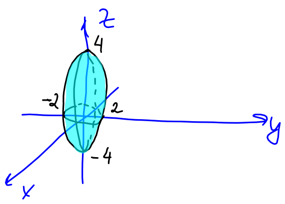
Example 4. Find the domain of the function $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$.

Domain: $16-4x^2-4y^2-z^2>0$

$4x^2+4y^2+z^2<16$

$\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}<1$ inside of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{4}+\frac{z^2}{16}=1$



Example 5. Describe the level surfaces of the function $f(x, y, z) = x^2 - y^2 + z^2$.

level surfaces : $k = x^2 - y^2 + z^2$

if $k=0$ — a cone

if $k>0$ — a hyperboloid of one sheet

if $k<0$ — a hyperboloid of two sheets

A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots x_n)$ to an n-tuple $(x_1, x_2, \ldots x_n)$ of real numbers. The notation

$$f : D \subset \mathbb{R}^n \to \mathbb{R}$$

is used to signify that f is a real valued function whose domain D is a subset of $\mathbb{R}^n$.