Chapter 12. **Partial derivatives.**
Section 12.3 **Partial derivatives.**

**Definition.** If $f$ is a function of two variables, its **partial derivatives** are the functions $f_x$ and $f_y$ defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

**Notation for partial derivatives:** If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial z}{\partial x} \leftarrow$$ partial derivative of $f$ with respect to $x$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \frac{\partial z}{\partial y} \leftarrow$$ partial derivative of $f$ with respect to $y$.

**Rule for finding partial derivatives of $z = f(x,y)$:**

1. To find $f_x$, regards $y$ as a constant and differentiate $f(x,y)$ with respect to $x$.
2. To find $f_y$, regards $x$ as a constant and differentiate $f(x,y)$ with respect to $y$.

**Example 1.** Find the first partial derivatives of the following functions:
(a) $f(x,y) = x^4 + x^2y^2 + y^4$

$$\frac{\partial f}{\partial x} = 4x^3 + 2xy^2 + 0$$
$$\frac{\partial f}{\partial y} = 0 + 2yx^2 + 4y^3$$

(b) $f(x,y) = x^y$

$(x^n)' = nx^{n-1}$

$$\boxed{\frac{\partial f}{\partial x} = f_x = y\,x^{y-1}}$$ $y$ is a "constant"

$x$ is a "constant" $(a^x)' = a^x \ln a$

$$\boxed{\frac{\partial f}{\partial y} = f_y = x^y \ln x}$$

(c) $f(x,y) = e^x \tan(x-y)$

Product Rule

$$f_x = e^x \tan(x-y) + e^x \sec^2(x-y)\,\cancel{(x-y)}_x^{\,1}$$

$$= \boxed{e^x \tan(x-y) + e^x \sec^2(x-y)}$$

$$f_y = e^x \sec^2(x-y)\,\cancel{(x-y)}_y^{\,-1}$$

$$= \boxed{-e^x \sec^2(x-y)}$$

$$\frac{\partial z}{\partial x} = z_x = z'_x$$

**Example 2.** Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z$ is define implicitly as a function of $x$ and $y$ by the equation

$$xyz = \cos(x+y+z)$$

Chain Rule

$\dfrac{\partial z}{\partial x}$:

$$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}\left(\cos(x+y+z)\right)$$

$$\cancel{\frac{\partial}{\partial x}(x)}^{\,1}\, yz + xy\frac{\partial z}{\partial x} = -\sin(x+y+z)\,\cancel{\frac{\partial}{\partial x}(x+y+z)}^{\,1+\frac{\partial z}{\partial x}}$$

$$yz + xy\frac{\partial z}{\partial x} = -\left(1+\frac{\partial z}{\partial x}\right)\sin(x+y+z)$$

Solve for $\dfrac{\partial z}{\partial x}$:

$$yz + xy\frac{\partial z}{\partial x} = -\sin(x+y+z) - \frac{\partial z}{\partial x}\sin(x+y+z)$$

$$xy\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\sin(x+y+z) = -\sin(x+y+z) - yz$$

$$\frac{\partial z}{\partial x}\left(xy + \sin(x+y+z)\right) = -\sin(x+y+z) - yz$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\sin(x+y+z) - yz}{xy + \sin(x+y+z)}}$$

$\dfrac{\partial z}{\partial y}$:

$$\frac{\partial}{\partial y}(xyz) = \frac{\partial}{\partial y}\left(\cos(x+y+z)\right)$$

$$xz + xy\frac{\partial z}{\partial y} = -\sin(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

Solve for $\dfrac{\partial z}{\partial y}$.

$$\boxed{\frac{\partial z}{\partial y} = \frac{-\sin(x+y+z) - xz}{xy + \sin(x+y+z)}}$$

**Functions of more that two variables.** Partial derivatives can also be defined for functions of three or more variables.

If $f$ is a function of three variables $x$, $y$, and $z$, ten its partial derivative with respect to $x$ can be defined as

$$\frac{\partial f}{\partial x} = f_x(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

and it is found by regarding $y$ and $z$ as constants and differentiating $f(x,y,z)$ with respect to $x$.

In general, if $u$ is function of $n$ variables, $u = f(x_1, x_2, ..., x_n)$, its partial derivative with respect to $x_i$ is

$$\frac{\partial u}{\partial x_i} = f_{x_i}(x_1, x_2, ..., x_n) = \lim_{h\to 0} \frac{f(x_1, ..., x_i + h, , , x_n) - f(x_1, ..., x_i, , , x_n)}{h}$$

**Higher derivatives.** If $z = f(x,y)$, then its **second partial derivatives** are defined as

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

**Clairaut's Theorem** Suppose $f$ is defined on a disk $D$ that contains the point $(a,b)$. If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$, then

$$f_{xy}(a,b) = f_{yx}(a,b). \qquad \boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

**Example 3.** Find all the second partial derivatives for the function $f(x,y) = (x^2+y^2)^{3/2}$

$$f(x,y) = (x^2+y^2)^{3/2}$$

$$\frac{\partial f}{\partial x} = \frac{3}{2}(x^2+y^2)^{1/2}\, \cancel{\frac{\partial}{\partial x}(x^2+y^2)}^{\,2x}$$
$$= \frac{3}{\cancel{2}}(x^2+y^2)^{1/2}(\cancel{2}x)$$
$$= 3x(x^2+y^2)^{1/2}$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}(x^2+y^2)^{1/2}\, \cancel{\frac{\partial}{\partial y}(x^2+y^2)}^{\,2y}$$
$$= \frac{3}{\cancel{2}}(x^2+y^2)^{1/2}(\cancel{2}y)$$
$$= 3y(x^2+y^2)^{1/2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x(x^2+y^2)^{1/2}\right)$$
$$= 3(x^2+y^2)^{1/2} + 3x\,\frac{1}{2}(x^2+y^2)^{-1/2}\, \cancel{\frac{\partial}{\partial x}(x^2+y^2)}^{\,2x}$$
$$= \boxed{3(x^2+y^2)^{1/2} + \frac{3}{2}x(x^2+y^2)^{-1/2}(2x)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3y(x^2+y^2)^{1/2}\right)$$
$$= 3y\,\frac{1}{2}(x^2+y^2)^{-1/2}\, \cancel{\frac{\partial}{\partial y}(x^2+y^2)}^{\,2x}$$
$$= \boxed{3xy(x^2+y^2)^{-1/2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left(3y(x^2+y^2)^{1/2}\right)$$
$$= 3(x^2+y^2)^{1/2} + 3y\,\frac{1}{2}(x^2+y^2)^{-1/2}(2y)$$
$$= \boxed{3(x^2+y^2)^{1/2} + 3y^2(x^2+y^2)^{-1/2}}$$

**Example 4.** Determine whether the function $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u = e^{-x}\cos y - e^{-y}\cos x$$

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y + e^{-y}\sin x \qquad \frac{\partial u}{\partial y} = -e^{-x}\sin y + e^{-y}\cos x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(-e^{-x}\cos y + e^{-y}\sin x\right) \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-e^{-x}\sin y + e^{-y}\cos x\right)$$

$$= e^{-x}\cos y + e^{-y}\cos x \qquad = -e^{-x}\cos y - e^{-y}\cos x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\cancel{e^{-x}\cos y} + \cancel{e^{-y}\cos x}\right) + \left(\cancel{-e^{-x}\cos y} - \cancel{e^{-y}\cos x}\right)$$

$$= 0 \qquad \boxed{\text{YES}}$$

**Example 5.** Find $f_{xyz}$ for the function $f(x, y, z) = e^{xyz}$.

$$\frac{\partial^3 f}{\partial x \partial y \partial z}$$

$$\frac{\partial}{\partial x}(xyz)$$

$$\frac{\partial f}{\partial x} = yz\, e^{xyz}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(yz\, e^{xyz}\right)$$

$$\frac{\partial}{\partial y}(xyz)$$

$$= z e^{xyz} + yz e^{xyz}(xz)$$

$$= z e^{xyz} + xyz^2 e^{xyz}$$

$$= (z + xyz^2) e^{xyz}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial z}\left((z + xyz^2) e^{xyz}\right)$$

$$\frac{\partial}{\partial z}(xyz)$$

$$= (1 + 2xyz) e^{xyz} + (z + xyz^2) e^{xyz}(xy)$$