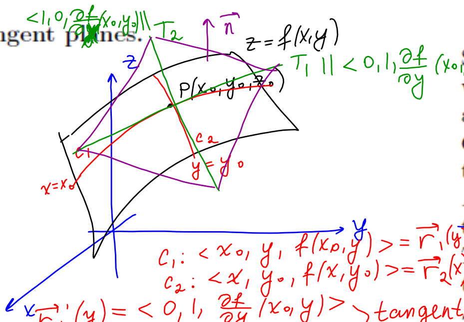
Chapter 12. **Partial derivatives.**
Section 12.4 **Tangent planes and differentials.**

Tangent planes.



Suppose a surface S has equation $z = f(x,y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. P lies on both C_1 and C_2. ~~Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2~~ Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. The **tangent plane** to the surface S at the point P is defined to be the plane that containts both of the tangent lines T_1 and T_2.

$C_1: \langle x_0, y, f(x_0,y)\rangle = \vec{r}_1(y)$

$C_2: \langle x, y_0, f(x,y_0)\rangle = \vec{r}_2(x)$

$\vec{r}_1\,'(y) = \langle 0, 1, \frac{\partial f}{\partial y}(x_0,y)\rangle$ tangent vectors

$\vec{r}_2\,'(x) = \langle 1, 0, \frac{\partial f}{\partial x}(x,y_0)\rangle$

$\boxed{\vec{n} = \vec{r}_1\,'(y_0) \times \vec{r}_2\,'(x_0)} = \langle 0, 1, \frac{\partial f}{\partial y}(x_0,y_0)\rangle \times \langle 1, 0, \frac{\partial f}{\partial x}(x_0,y_0)\rangle = \langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1\rangle$

An equation of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0,y_0,z_0)$ is

$$\boxed{z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)}$$

Example 1. Find the equation of the tangent plane to the surface $z = \ln(2x + y)$ at the point $(-1, 3, 0)$.

$x_0 = -1$
$y_0 = 3$
$z_0 = 0$

$\frac{\partial z}{\partial x} = \frac{2}{2x+y}$; $\frac{\partial z}{\partial x}(-1,3) = \frac{2}{1} = 2$

$\frac{\partial z}{\partial y} = \frac{1}{2x+y}$; $\frac{\partial z}{\partial y}(-1,3) = 1$

$\vec{n} = \langle \frac{\partial z}{\partial x}(-1,3), \frac{\partial z}{\partial y}(-1,3), -1\rangle$
$= \langle 2, 1, -1\rangle$

Equation of tangent plane is $\boxed{z = 2(x+1) + 1(y-3)}$

Differentials. Consider a function of two variables $z = f(x,y)$. If x and y are given increments Δx and Δy, then the corresponding **increment** of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y)$$ increment of z

The increment Δz represents the change in the value of f when (x,y) changes to $(x + \Delta x, y + \Delta y)$.

The **differentials** dx and dy are independent variables. The **differential** dz (or the **total differential**), is defined by

$$\boxed{dz = f_x(x,y)dx + f_y(x,y)dy}$$ differential of the function $z(x,y)$

Example 2. Find the differential of the function $z = \sqrt[3]{x + y^2}$.

$= (x+y^2)^{1/3}$

$\frac{\partial z}{\partial x} = \frac{1}{3}(x+y^2)^{-2/3}$

$\frac{\partial z}{\partial y} = \frac{1}{3}(x+y^2)^{-2/3}(2y)$

$dz = \frac{1}{3}(x+y^2)^{-2/3}\underline{dx} + \frac{1}{3}(x+y^2)^{-2/3}(2y)\underline{dy}$

If we take
$$dx = \Delta x = x - a \quad dy = \Delta y = y - b$$
then the differential of z is
$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
On the other hand, the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$ is
$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
We see that dz represents the change of height of the tangent plane whereas Δz represent the change in height of the surface $z = f(x,y)$ when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.
If $dx = \Delta x$ and $dy = \Delta y$ are small, then $\Delta z \approx dz$ and
$$\boxed{f(a+\Delta x, b+\Delta y) \approx f(a,b) + dz}$$
$$\boxed{= f(a,b) + \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y}$$

Example 3.

1. Use differential to approximate the value of the function $f(x,y) = \sqrt{20 - x^2 - 7y^2}$ at the point $(1.95, 1.08)$.

$$f(1.95, 1.08) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y$$

$a = 2 \qquad \Delta x = -0.05$

$b = 1 \qquad \Delta y = 0.08$

$$f(2,1) = \sqrt{20-4-7} = \sqrt{9} = 3$$
$$\frac{\partial f}{\partial x} = \frac{1}{2}(20-x^2-7y^2)^{-1/2}(-2x) = -x(20-x^2-7y^2)^{-1/2}$$
$$\frac{\partial f}{\partial x}(2,1) = -2(20-4-7)^{-1/2} = -\frac{2}{3}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2}(20-x^2-7y^2)^{-1/2}(-14y) = -7y(20-x^2-7y^2)^{-1/2}$$
$$\frac{\partial f}{\partial y}(2,1) = -\frac{7}{3}$$
$$\boxed{f(1.95, 1.08) \approx 3 + \left(-\frac{2}{3}\right)(-0.05) + \left(-\frac{7}{3}\right)(0.08)}$$

2. Use differential to approximate the number $(\sqrt{99} + \sqrt[3]{124})^2$

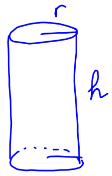
$$f(x,y) = (\sqrt{x} + \sqrt[3]{y})^2 = (x^{1/2} + y^{1/3})^2$$

$a = 100, \quad \Delta x = -1$

$b = 125, \quad \Delta y = -1$

$$(\sqrt{99} + \sqrt[3]{124})^2 \approx f(100,125) + \frac{\partial f}{\partial x}(100,125)\Delta x + \frac{\partial f}{\partial y}(100,125)\Delta y$$
$$f(100,125) = (\sqrt{100} + \sqrt[3]{125})^2 = 225$$
$$\frac{\partial f}{\partial x} = 2(x^{1/2} + y^{1/3})\frac{1}{2}x^{-1/2}; \quad \frac{\partial f}{\partial x}(100,125) = 2(15)\frac{1}{2\cdot 10} = \frac{15}{10} = \frac{3}{2}$$
$$\frac{\partial f}{\partial y} = 2(x^{1/2} + y^{1/3})\frac{1}{3}y^{-2/3}; \quad \frac{\partial f}{\partial y}(100,125) = 2(15)\frac{1}{3\cdot 25} = \frac{2}{5}$$
$$\boxed{(\sqrt{99} + \sqrt[3]{124})^2 \approx 225 + \frac{3}{2}(-1) + \frac{2}{5}(-1)}$$

Example 4. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.



$SA = 2\pi r^2 + 2\pi r h$, $SA(4,12) = 2\pi(16) + 2\pi(4)(12)$

$\Delta r = \Delta h = 0.04$

$r_0 = 4$, $h_0 = 12$

Error of approximation:

$\Delta(SA) \approx \frac{\partial(SA)}{\partial r}(r_0,h_0)\Delta r + \frac{\partial(SA)}{\partial h}(r_0,h_0)\Delta h$

$\frac{\partial(SA)}{\partial r} = 4\pi r + 2\pi h$; $\frac{\partial(SA)}{\partial r}(4,12) = 16\pi + 24\pi = 40\pi$

$\frac{\partial(SA)}{\partial h} = 2\pi r$; $\frac{\partial(SA)}{\partial h}(4,12) = 8\pi$

$\Delta(SA) \approx (40\pi + 8\pi)\, 0.04 = 48\pi(0.04)$

Amount of tin $= SA(4,12) + \Delta(SA)$

$= \boxed{(32\pi + 96\pi) + (48\pi)(0.04)}$

Functions of three or more variables. If $u = f(x,y,z)$, then the **increment** of u is

$$\Delta u = f_x(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$

The **differential** du is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$du = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz$$

If $dx = \Delta x$, $dy = \Delta y$, and $dz = \Delta z$ are small and f has continuous partial derivatives, then $\Delta u \approx du$.