

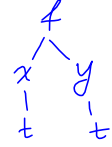
Section 12.5 The chain rule.

For a function of a single variable $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$.

$$z = f(x, y) = f(x(t), y(t))$$

The Chain Rule (case 1). Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$



Example 1. If $z = x^2y^3 + x^3y^2$, where $x = 1 + \sqrt{t}$ and $y = 1 + e^{2t}$, find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = 2xy^3 + 3x^2y^2$$

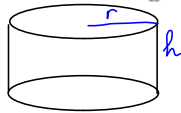
$$\frac{\partial f}{\partial y} = 3x^2y^2 + 2x^3y$$

$$\frac{dx}{dt} = \frac{1}{2} t^{-1/2}$$

$$\frac{dy}{dt} = 2e^{2t}$$

$$\frac{dz}{dt} = (2xy^3 + 3x^2y^2) \frac{1}{2} t^{-1/2} + (3x^2y^2 + 2x^3y) 2e^{2t}$$

Example 2. The radius of a right circular cylinder is decreasing at a rate of 1.2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 150 cm.



$$V = \pi r^2 h, \quad r = r(t), \quad h = h(t)$$

$$\frac{dV}{dt} \text{ ? when } r = 80, \quad h = 150$$

$$\frac{dr}{dt} = -1.2, \quad \frac{dh}{dt} = 3$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt}$$

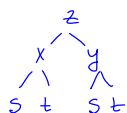
$$\frac{\partial V}{\partial r} = 2\pi r h, \quad \frac{\partial V}{\partial h} = \pi r^2$$

$$\frac{\partial V}{\partial r} (80, 150) = 2\pi(80)(150)$$

$$\frac{\partial V}{\partial h} (80, 150) = \pi(80)^2$$

$$\frac{dV}{dt} = 2\pi(80)(150)(-1.2) + \pi(80)^2(3)$$

The Chain Rule (case 2). Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$, $y = h(s, t)$, and the partial derivatives g_s , g_t , h_s , and h_t exist. Then



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

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Example 3. If $z = x^2 \sin y$, where $x = s^2 + t^2$ and $y = 2st$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$. $y = 2st, \frac{\partial y}{\partial s} = 2t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$= 2x \sin y (2s) + x^2 \cos y (2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

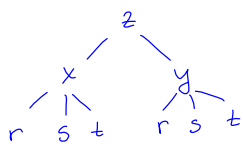
$$= 2x \sin y (2t) + x^2 \cos y (2s)$$

The Chain Rule (general version). Suppose that u is a differentiable function of the n variables x_1, x_2, \dots, x_n and each x_j is a function of the m variables t_1, t_2, \dots, t_m such that all partial derivatives $\partial x_j / \partial t_i$ exist ($j = 1, 2, \dots, n, i = 1, 2, \dots, m$). Then u is a function of t_1, t_2, \dots, t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \dots, m$.

Example 4. If $z = \frac{x}{y}$, where $x = re^{st}$, $y = rse^t$, find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$.



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$$

$$= \frac{1}{y} e^{st} - \frac{x}{y^2} se^t$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$= \frac{1}{y} tre^{st} - \frac{x}{y^2} re^t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$= \frac{1}{y} sre^{st} - \frac{x}{y^2} rse^t$$

Implicit differentiation. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x , that is $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f .

In order to find dy/dx , we differentiate both parts of the equation $F(x, y) = 0$ with respect to x :

$$\frac{d}{dx}(F(x, y)) = \frac{d}{dx}(0)$$

$$\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

Since $\frac{dx}{dx} = 1$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Example 5. Find dy/dx if $x^2 - xy + y^3 = 8$.

$$F(x, y) = x^2 - xy + y^3 - 8$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{2x - y}{-x + 3y^2}$$

Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. If F is differentiable and f_x and f_y exist, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows

$$\frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0 \quad z = z(x, y)$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

or

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example 6. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xyz = \cos(x + y + z)$.

$$F(x, y, z) = xyz - \cos(x + y + z)$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{yz + \sin(x + y + z)}{xy + \sin(x + y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{xz + \sin(x + y + z)}{xy + \sin(x + y + z)}$$