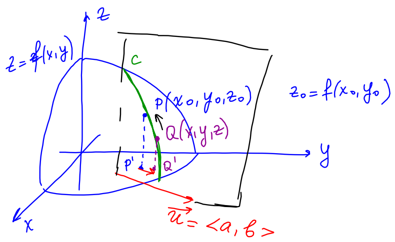
Chapter 12. **Partial derivatives.**

Section 12.6 **Directional derivatives and the gradient vector.**

Let $z = f(x, y)$. We wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\vec{u} =< a, b >$.



To do this we consider the surface S with equation $z = f(x, y)$ and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction $\vec{u}$ intersects S in a curve C. The slope of the tangent line T to C at P is the rate of change ~~of change~~ of z in the direction of $\vec{u}$.

Let $Q(x, y, z)$ be another point on C. If $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$ are projections of P and Q on the xy-plane, then the vector $\overrightarrow{P'Q'} =< x - x_0, y - y_0, 0 >$ is parallel to $\vec{u}$ and so

$$\boxed{\overrightarrow{P'Q'} = h\vec{u} =< ha, hb >}$$

for some scalar h. Therefore

$$x - x_0 = ha \quad y - y_0 = hb$$

and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z in the direction of $\vec{u}$, which is called the directional derivative of f in the direction of $\vec{u}$.

$P'(x_0, y_0, 0)$ is the projection of $P(x_0, y_0, z_0)$ onto the (xy)-plane

$Q'(x, y, 0)$ is the projection of $Q(x, y, z)$ onto the (xy)-plane

$\overrightarrow{P'Q'}$ is parallel to $\vec{u}$.

$\overrightarrow{P'Q'} = \langle x - x_0, y - y_0, 0 \rangle$

Definition. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} =< a, b >$ is

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{z - z_0}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}}$$

if this limit exists.

Theorem. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} =< a, b >$ and

$$\boxed{D_{\vec{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y) a + \frac{\partial f}{\partial y}(x, y) b}$$

If the unit vector $\vec{u}$ makes an angle θ with the positive x-axis, then

$$\boxed{D_{\vec{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y) \cos\theta + \frac{\partial f}{\partial y}(x, y) \sin\theta}$$

Example 1. Find the directional derivative of the function $f(x, y) = y^x$ at the point $(1, 2)$ in the direction of the unit vector $\vec{u}$ given by angle $\theta = -2\pi/3$.

$$D_{\vec{u}} f = \frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta$$

$$D_{\vec{u}} f(1,2) = \frac{\partial f}{\partial x}(1,2) \cos\left(-\frac{2\pi}{3}\right)^{-1/2} + \frac{\partial f}{\partial y}(1,2) \sin\left(-\frac{2\pi}{3}\right)^{-\frac{\sqrt{3}}{2}}$$

$\frac{\partial f}{\partial x} = y^x \ln y$	$\frac{\partial f}{\partial x}(1,2) = 2 \ln 2$
$\frac{\partial f}{\partial y} = x y^{x-1}$	$\frac{\partial f}{\partial y}(1,2) = 2^{1-1} = 1$

$$\boxed{D_{\vec{u}} f(1,2) = 2 \ln 2 \left(-\frac{1}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)}$$

Definition. If f is a function of two variables x and y, then the gradient of f is defined by

$$\boxed{\operatorname{grad}(f) = \nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle.}$$

Then

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}.}$$

Example 2. Find the directional derivative of the function $f(x,y) = xe^{xy}$ at the point $(-3,0)$ in the direction of the vector $\vec{v} = 2\vec{\imath} + 3\vec{\jmath}$.

$= <2,3>, \quad |\vec{v}| = \sqrt{4+9} = \sqrt{13}$

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = < \dfrac{2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}} >$

$\nabla f(x,y) = < e^{xy} + xye^{xy}, \ x^2 e^{xy} >$

$\nabla f(-3,0) = < 1, 9 >$

$\boxed{D_{\vec{u}} f(-3,0) = (1)\dfrac{2}{\sqrt{13}} + 9\left(\dfrac{3}{\sqrt{13}}\right)}$

For a function of three variables $w = f(x,y,z)$ the gradient vector is

$$\boxed{\operatorname{grad}(f) = \nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z) \right\rangle}$$

and

$$\boxed{D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}.}$$

Theorem. Suppose f is a differentiable function of two or three variables and $\vec{x} =< x, y >$ if f is a function of two variables $\vec{x} =< x, y, z >$ if f is a function of three variables. The maximum value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

Example 3. Suppose that over a certain region of space the electrical potential V is given by $V(x,y,z) = 5x^2 - 3xy + xyz$.

1. Find the rate of change of the potential at $P(3,4,5)$ in the direction of the vector $\vec{v} =< 1, 1, -1 >$.

$|\vec{v}| = \sqrt{1+1+1} = \sqrt{3}$

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = < \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}} >$

$\nabla V = < \dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}, \dfrac{\partial V}{\partial z} >$

$= < 10x - 3y + yz, \ -3x + xz, \ xy >$

$\nabla V(3,4,5) = < 30 - 12 + 20, \ -9 + 15, \ 12 >$

$= < 38, 6, 12 >$

$D_{\vec{u}} V = < 38, 6, 12 > \cdot < \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}} >$

$= \boxed{\dfrac{32}{\sqrt{3}}}$

2. In which direction does V change most rapidly at P?

$$\boxed{\nabla V(3,4,5) = \langle 38, 6, 12 \rangle}$$

3. What is the maximum rate of change at P?

$$|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2}$$

$$= \boxed{\sqrt{1624}}$$

Tangent planes to level surfaces.

Suppose S is a surface with equation $F(x,y,z)=k$, that is, it is a level surface of the function $w=F(x,y,z)$, and let $P(x_0,y_0,z_0)$ be a point on S.

We define the tangent plane to the level surface $F(x,y,z)=k$ at $P(x_0,y_0,z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0,y_0,z_0)$ and its equation is

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0.}$$

The normal line to S at P is the line passing through P and perpendicular to the tangent plane (its direction is given by the gradient vector $\nabla F(x_0,y_0,z_0)$). Its symmetric equations are

$$\boxed{\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}}$$

If the equation of a surface S is of the form $z=f(x,y)$, we can rewrite

$$F(x,y,z)=f(x,y)-z=0$$

and regard S as a level surface of F with $k=0$. Then

$$F_x(x_0,y_0,z_0)=f_x(x_0,y_0), \quad F_y(x_0,y_0,z_0)=f_y(x_0,y_0), \quad F_z(x_0,y_0,z_0)=-1$$

so the equation of the tangent plane to S at (x_0,y_0) is

$$\boxed{f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0}$$

and the equation of the normal line to S at (x_0,y_0) is

$$\boxed{\frac{x-x_0}{f_x(x_0,y_0)}=\frac{y-y_0}{f_y(x_0,y_0)}=\frac{z-z_0}{-1}}$$

Example 4. Find equations of the tangent plane and the normal line to the surface $x^2-2y^2-3z^2+xyz=4$ at the point $(3,-2,-1)$.

$$x^2-2y^2-3z^2+xyz=4$$

$$F(x,y,z)=x^2-2y^2-3z^2+xyz-4$$

$$\nabla F = \langle 2x+yz, -4y+xz, -6z+xy \rangle$$

$$\nabla F(3,-2,-1) = \langle 8, 5, 0 \rangle$$

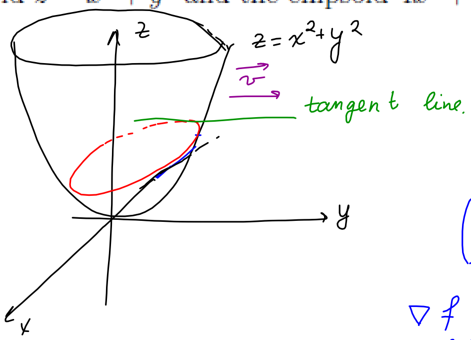
Equation of the tangent plane:

$$\boxed{8(x-3)+5(y+2)+0(z+1)=0}$$

$$\boxed{8x+5y-14=0}$$

Equations of the normal line: $\boxed{\begin{cases} \dfrac{x-3}{8} = \dfrac{y+2}{5} \\ z=-1 \end{cases}}$

Example 5. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.



$f(x, y, z) = x^2 + y^2 - z$

$g(x, y, z) = 4x^2 + y^2 + z^2 - 9$

The tangent line is perpendicular to both $\nabla f(-1, 1, 2)$ and $\nabla g(-1, 1, 2)$

$$\boxed{\vec{v} = \nabla f(-1,1,2) \times \nabla g(-1,1,2)}$$

$\nabla f = \langle 2x, 2y, -1 \rangle$ | $\nabla g = \langle 8x, 2y, 2z \rangle$

$\nabla f(-1,1,2) = \langle -2, 2, -1 \rangle$ | $\nabla g(-1,1,2) = \langle -8, 2, 4 \rangle$

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ -8 & 2 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & -1 \\ -8 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 2 \\ -8 & 2 \end{vmatrix}$$

$$= 10\vec{i} + 16\vec{j} + 12\vec{k}$$

Parametric equations of the tangent line:

$$\boxed{\begin{aligned} x &= -1 + 10t \\ y &= 1 + 16t \\ z &= 2 + 12t \end{aligned}}$$