Chapter 12. **Partial derivatives.**
Section 12.7 **Maximum and minimum values.**

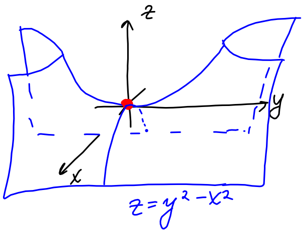
Definition. A function of two variables has a **local maximum at** (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a **local maximum value.** If $f(x, y) \geq f(a, b)$ for all (x, y) in such a disk, $f(a, b)$ is a **local minimum value.**

Theorem. If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = f_y(a, b) = 0$.

Geometric interpretation of the Theorem: if the graph of f has a tangent plane at a local extremum, then the tangent plane must be horizontal.

A point (a, b) such that $f_x(a, b) = f_y(a, b) = 0$ or one of these partial derivatives does not exist, is called a **critical point of** f. At a critical point, a function could have a local minimum or a local maximum or neither.

Example 1. Find the extreme values of $f(x, y) = y^2 - x^2$.



$$\frac{\partial f}{\partial x}(x,y) = -2x = 0 \qquad \begin{cases} -2x = 0 \\ 2y = 0 \end{cases} \qquad \begin{array}{l} x = 0 \\ y = 0 \end{array}$$

$$\frac{\partial f}{\partial y} = 2y = 0$$

$(0,0)$ is a critical point.

$f(0,0) = 0$

$f(x,0) = -x^2 \leq 0$ $\qquad$ $f(x,0) \leq f(0,0)$ for all x.

$f(0,y) = y^2 \geq 0$ $\qquad$ $f(0,y) \geq f(0,0)$ for all y.

$f(0,0)$ is neither local min nor local max

it is the saddle point.

Second derivative test. Suppose the second partial derivatives of f are continuous in a disk with center (a, b), and suppose $f_x(a, b) = f_y(a, b) = 0$. Let

$$\boxed{D = D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local extremum. $f(a, b)$ is a saddle point.
If $D = 0$ the test gives no information.

Example 2. Find the local extrema of $f(x, y) = x^3 - 3xy + y^3$.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0 \qquad \begin{cases} x^2 - y = 0 \\ -x + y^2 = 0 \end{cases} \qquad y = x^2$$

$$\frac{\partial f}{\partial y} = -3x + 3y^2 = 0$$

$-x + x^4 = 0$

$x(x^3 - 1) = 0$

$x(x-1)(x^2+x+1) = 0$

$x_1 = 0 \qquad x_2 = 1$

$y_1 = 0 \qquad y_2 = 1$

Critical points $(0,0)$ and $(1,1)$.

The 2nd derivative test.

	$(0,0)$	$(1,1)$
$\frac{\partial^2 f}{\partial x^2} = 6x$	$\frac{\partial^2 f}{\partial x^2}(0,0) = 0$	$\frac{\partial^2 f}{\partial x^2}(1,1) = \boxed{6>0}$
$\frac{\partial^2 f}{\partial x \partial y} = -3$	$\frac{\partial^2 f}{\partial x \partial y}(0,0) = -3$	$\frac{\partial^2 f}{\partial x \partial y}(1,1) = -3$
$\frac{\partial^2 f}{\partial y^2} = 6y$	$\frac{\partial^2 f}{\partial y^2}(0,0) = 0$	$\frac{\partial^2 f}{\partial y^2}(1,1) = 6$

$$D(0,0) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0$$

$(0,0)$ is a saddle point.

$$D(1,1) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(1,1) & \frac{\partial^2 f}{\partial x \partial y}(1,1) \\ \frac{\partial^2 f}{\partial x \partial y}(1,1) & \frac{\partial^2 f}{\partial y^2}(1,1) \end{vmatrix} = \begin{vmatrix} \boxed{6>0} & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 = \boxed{27 > 0}$$

$(1,1)$ is a local min

Example 3. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

$P(x,y,z)$ is an arbitrary point in the surface.

$z^2 = xy+1$.

The distance from $(0,0,0)$ to P:

$$d = \sqrt{x^2+y^2+z^2}$$

$$= \sqrt{x^2+y^2+xy+1}$$

$$d^2 = x^2+y^2+xy+1 = f(x,y)$$

Find the local min of $f(x,y)$.

$$\frac{\partial f}{\partial x} = 2x+y = 0 \qquad \begin{cases} 2x+y=0 \\ 2y+x=0 \end{cases} \qquad y=-2x$$

$$\frac{\partial f}{\partial y} = 2y+x = 0 \qquad 2(-2x)+x=0$$

$$-3x=0$$

$$x=0,\ y=0.$$

Critical point $(0,0)$.

The 2nd derivative test.

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad D(0,0) = \begin{vmatrix} \boxed{2>0} & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = \boxed{3>0}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$(0,0)$ is a local min.

$$z^2 = xy+1$$

@ $(0,0)$: $z^2 = (0)(0)+1$

$$= 1$$

$$z = \pm 1$$

$$\boxed{(0,0,\pm 1)}$$

Absolute maximum and minimum values. **A closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. A **bounded set** in $\mathbb{R}^2$ is one that contained within some disk.

Extreme value theorem for functions of two variables. If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values of f from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

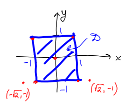
2

Example 4. Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 + x^2y + 4$ on the set $D = \{(x,y) \| |x| \le 1, |y| \le 1\}$.

$f(x,y) = x^2+y^2+x^2y+4$

$D = \{(x,y) \mid |x| \le 1,\ |y| \le 1\}$

$-1 \le x \le 1,\ -1 \le y \le 1$



The region D is bounded by the lines $x=1,\ x=-1,\ y=1,\ y=-1$

Step 1. Critical values of f in D. $(-1<x<1, -1<y<1)$

$$\frac{\partial f}{\partial x} = 2x+2xy = 0 \qquad \begin{cases} 2x+2xy=0 \\ 2y+x^2=0 \end{cases} \qquad 2x(1+y)=0$$

$$\frac{\partial f}{\partial y} = 2y+x^2 = 0 \qquad 2y=-x^2 \qquad x=0 \text{ or } y=-1$$

$$y=0 \qquad x^2=2$$

$$x=\pm\sqrt{2}$$

$(0,0)$, $\cancel{(\pm\sqrt{2},-1)}$ not in D.

$f(0,0) = \boxed{4}$ absolute min value

Step 2. Critical values of f on the boundary.

$f(1,1) = 1+1+1+4 = \boxed{7}$ absolute max value

$f(-1,1) = 1+1+1+4 = \boxed{7}$

$f(1,-1) = 1+1-1+4 = \boxed{5}$

$f(-1,-1) = 1+1-1+4 = \boxed{5}$

$x=\pm 1,\ -1<y<1$

$f(x,y) = x^2+y^2+x^2y+4$

$g(y) = f(\pm 1, y) = 1+y^2+y+4$

$= y^2+y+5$

$g'(y) = 2y+1 = 0$

$y = -1/2 \qquad (\pm 1, -1/2)$

$f(\pm 1, -1/2) = 1 + \frac{1}{4} - \frac{1}{2} + 4 = \boxed{\frac{19}{4}}$

$y=1,\ -1<x<1$

$g(x) = f(x,1) = x^2+1+x^2+4$

$= 2x^2+5$

$g'(x) = 4x = 0 \qquad x=0. \qquad (0,1)$

$f(0,1) = 0+1+0+4 = \boxed{5}$

$y=-1,\ -1<x<1$

$f(x,-1) = x^2+1-x^2+4$

$= \boxed{5}$ for all $-1<x<1$