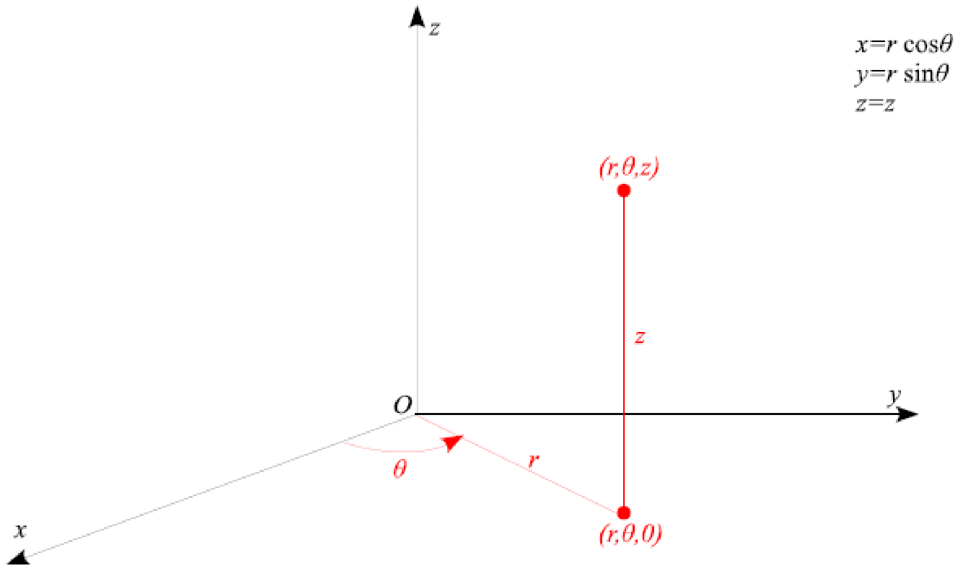
Section 13.10 **Triple integrals in cylindrical and spherical coordinates.**

Cylindrical coordinate system:



Suppose that E is a type 1 region whose projection D on the xy-plane is described in polar coordinates

$$E = \{(x,y,z) | (x,y) \in D, \varphi_1(x,y) \le z \le \varphi_2(x,y)\}$$

$$D = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Then

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)dz \right] dA$$

If we switch to cylindrical coordinates

$$\boxed{x = r\cos\theta,} \quad \boxed{y = r\sin\theta,} \quad \boxed{z = z,} \quad \boxed{dA = r\,dr\,d\theta}$$

we will get

$$\boxed{\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\varphi_1(r\cos\theta, r\sin\theta)}^{\varphi_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

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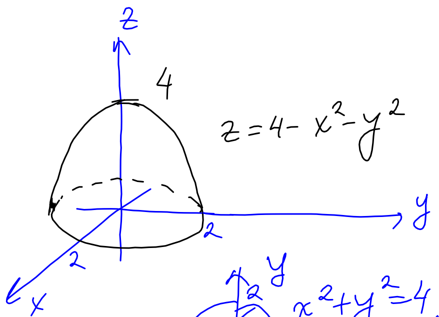
Example 1. Sketch the solid whose volume is given by $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\,dz\,dr\,d\theta$.

$0 \le \theta \le 2\pi$
$0 \le r \le 2$
$0 \le z \le 4 - r^2$

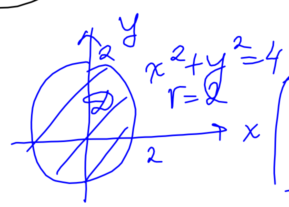
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$x^2 + y^2 = r^2$

$z = 0$ - the (xy)-plane
$z = 4 - r^2$ convert to rectangular coord.
$z = 4 - (x^2 + y^2)$ circular paraboloid.
vertex at $(0,0,4)$.

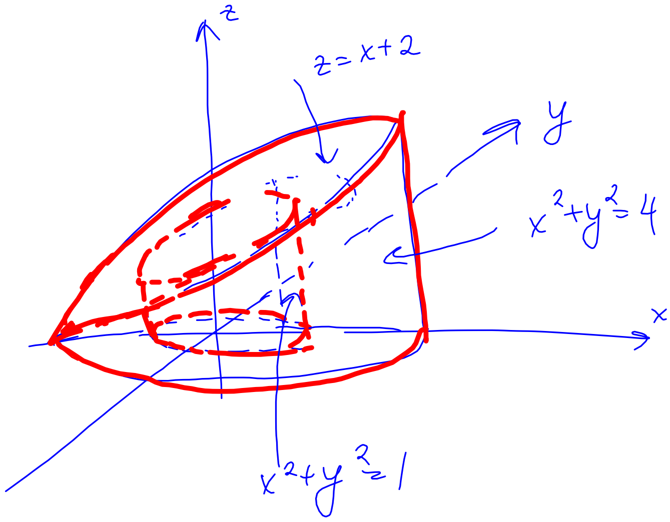


intersection of the paraboloid and the (xy)-plane
$4 - (x^2 + y^2) = 0$.
$x^2 + y^2 = 4$.



$$\boxed{\begin{array}{l} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array}}$$

Example 2. Evaluate $\iiint_E y\,dV$ where E is the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$, above the xy-plane, and below the plane $z=x+2$.



$0 \le z \le x+2$

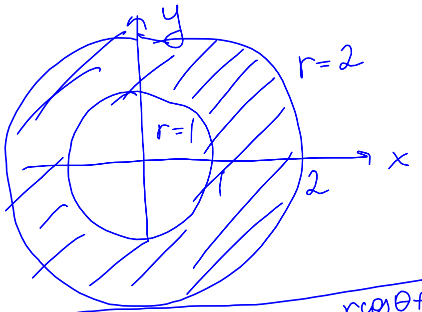
Cylindrical coord.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z.$

$dV = r\,dz\,dr\,d\theta$

$0 \le z \le x+2$
$0 \le z \le r\cos\theta + 2$
$1 \le r \le 2$
$0 \le \theta \le 2\pi$

Projection onto the (xy)-plane



$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r\sin\theta\, r\,dz\,dr\,d\theta$$

$\int_a^b\int_c^d f(x)g(y)\,dy\,dx = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta\, z\Big]_{z=0}^{z=r\cos\theta+2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta+2)\,dr\,d\theta$$

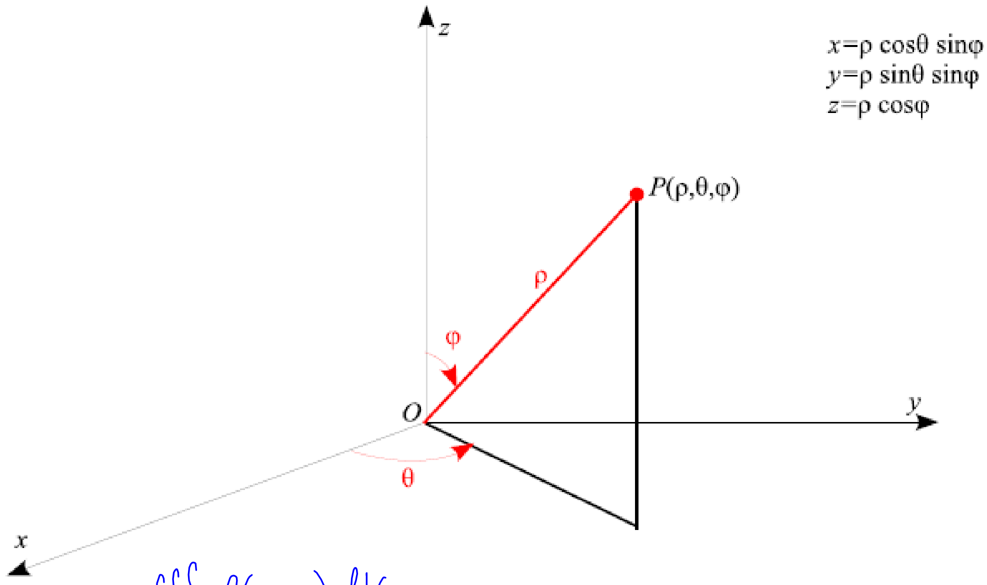
$$= \int_0^{2\pi}\int_1^2 r^3\sin\theta\cos\theta\,dr\,d\theta + 2\int_0^{2\pi}\int_1^2 r^2\sin\theta\,dr\,d\theta$$

$$= \underbrace{\int_0^{2\pi}\sin\theta\cos\theta\,d\theta}\int_1^2 r^3\,dr + 2\cancel{\int_0^{2\pi}\sin\theta\,d\theta}^{\,0}\int_1^2 r^2\,dr$$

$u = \sin\theta$
$du = \cos\theta\,d\theta$
$\theta = 0 \longrightarrow u = 0$
$\theta = 2\pi \longrightarrow u = 0$

$$= \int_0^0 u\,du \cdot \frac{r^4}{4}\Big]_1^2 = \boxed{0}$$

Spherical coordinate system:

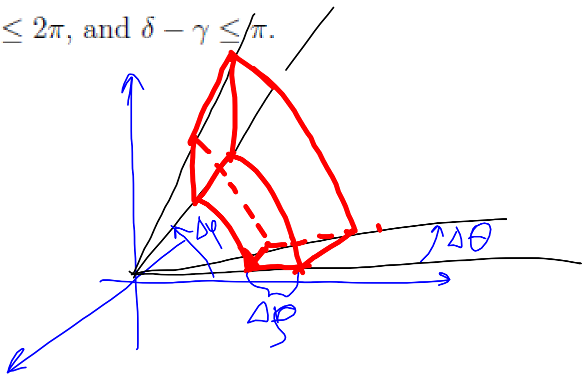


$\iiint\limits_E f(x,y,z)\,dV$

In this coordinate system the analog of rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \varphi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \varphi \leq \delta\}$$

where $a \geq 0$, $\beta - \alpha \leq 2\pi$, and $\delta - \gamma \leq \pi$.



$a \leq \rho \leq b$
$\alpha \leq \theta \leq \beta$
$\gamma \leq \varphi \leq \delta \quad (\delta - \gamma \leq \pi)$

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$$\boxed{\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_\gamma^\delta \int_a^b f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\boxed{\rho^2\sin\varphi}\,d\rho\,d\varphi\,d\theta}$$

To integrate over such region we consider a spherical partition P of E into smaller spherical wedges E_{ijk} by means of spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$ and half-cones $\varphi = \varphi_k$. The norm of P, $\|P\|$, is the length of the longest diagonal of these wedges. If $\|P\|$ is small, then E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho_i$, $\rho_i\Delta\varphi_k$ (arc of a circle with radius ρ_i, angle $\Delta\varphi_k$), and $\rho_i \sin\varphi_k \Delta\theta_j$ (arc of a circle with radius $\rho_i \sin\varphi_k$, angle $\Delta\theta_j$).

$$V(E_{ijk}) = \Delta V_{ijk} \approx \rho_i^2 \sin\varphi_j \Delta\rho_i \Delta\theta_j \Delta\varphi_k$$

Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of the point in E_{ijk}. Then there are exist ρ^*_{ijk}, θ^*_{ijk}, and φ^*_{ijk} such that

$$x^*_{ijk} = \rho^*_{ijk}\cos\theta^*_{ijk}\sin\varphi^*_{ijk}, \qquad y^*_{ijk} = \rho^*_{ijk}\sin\theta^*_{ijk}\sin\varphi^*_{ijk}, \qquad z^*_{ijk} = \rho^*_{ijk}\cos\varphi^*_{ijk}$$

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Then

$$\iiint_E f(x,y,z)dV = \lim_{\|P\|\to 0} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk} =$$

$$\lim_{\|P\|\to 0} f(\rho^*_{ijk}\cos\theta^*_{ijk}\sin\varphi^*_{ijk},\ \rho^*_{ijk}\sin\theta^*_{ijk}\sin\varphi^*_{ijk},\ \rho^*_{ijk}\cos\varphi^*_{ijk})[\rho^*_{ijk}]^2\sin\varphi^*_{ijk}\,\Delta\rho_i\,\Delta\theta_j\,\Delta\varphi_k =$$

Thus,

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_\gamma^\delta \int_a^b f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

where $E = \{(\rho,\theta,\varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$

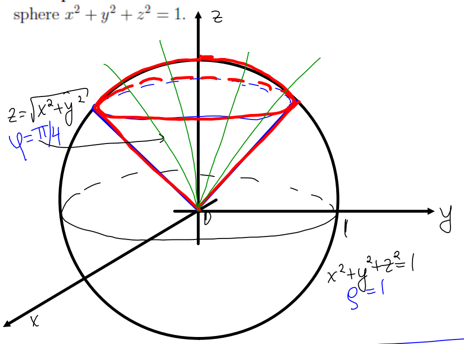
This formula can be extended to include more general spherical regions such as

$$E = \{(\rho,\theta,\varphi) | \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta, g_1(\theta,\varphi) \le \rho \le g_2(\theta,\varphi)\}$$

In this case

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_\gamma^\delta \int_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

Example 3. Find the volume of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$.



spherical coord.
$x=\rho\cos\theta\sin\varphi$
$y=\rho\sin\theta\sin\varphi$
$z=\rho\cos\varphi$
$x^2+y^2+z^2=\rho^2$
$dV=\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

Convert $z=\sqrt{x^2+y^2}$ into spherical coord.

$$\rho\cos\varphi = \sqrt{\rho^2\cos^2\theta\sin^2\varphi+\rho^2\sin^2\theta\sin^2\varphi}$$

$$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)}$$

$$\frac{\cancel{\rho}\cos\varphi = \cancel{\rho}\sin\varphi}{\cos\varphi}$$

$$\tan\varphi = 1$$

$$\boxed{\varphi = \pi/4}$$

origin $\le \rho \le$ sphere $0\le\rho\le 1$	z-axes $\le\varphi\le$ cone $0\le\varphi\le\pi/4$	$0\le\theta\le 2\pi$

$$V = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4}\sin\varphi\, d\varphi \cdot \int_0^1 \rho^2 d\rho$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_0^{\pi/4} \cdot \frac{\rho^3}{3}\Big]_0^1$$

$$= \boxed{\frac{2\pi}{3}\left[-\frac{\sqrt{2}}{2}+1\right]}$$