Section 13.1 **Double integrals over rectangles.**

We would like to define the double integral of a function f of two variables that is defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$$

We take a partition P of R into subrectangles. This is accomplished by partitioning the intervals $[a, b]$ and $[c, d]$ as follows:

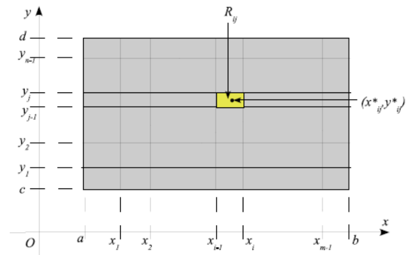
$$a = x_0 < x_1 < ... < x_{m-1} < x_m = b$$

$$c = y_0 < y_1 < ... < y_{n-1} < y_n = d$$

By drawing lines parallel to the coordinate axes through these partition points we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. There are mn of these subrectangles. If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, then the area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$.



Next we choose a point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$ and form the **double Riemann sum**

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A_{ij}$$

We denote by $\|P\|$ the **norm** of the partition, which is the length of the longest diagonal of all the subrectangles R_{ij}.

Definition. The **double integral** of f over the rectangle R is

$$\iint_R f(x, y) dA = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A_{ij}$$

if the limit exists.

Note 1. In view of the fact that $\Delta A_{ij} = \Delta x_i \Delta y_j$, another notation that is used sometimes for the double integral is

$$\iint_R f(x, y) dA = \iint_R f(x, y) dx\, dy$$

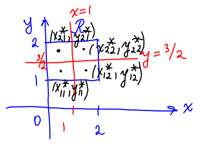
Note 2. A function f is called **integrable** if the limit in the definition exists.

Example 1. Find an approximation to the integral

$$\iint_R (x - 3y^2) dA$$

$0 \le x \le 2$ $\quad$ $1 \le y \le 2$

where $R = [0, 2] \times [1, 2]$, by computing the double Riemann sum with partition lines $x = 1$ and $y = 3/2$ and taking (x^*_{ij}, y^*_{ij}) to be the center of each rectangle.



$f(x,y) = x - 3y^2$

x^*_{11} is the midpoint of $[0,1]$

$x^*_{11} = 1/2$

y^*_{11} is the midpoint of $[1, 3/2]$

$y^*_{11} = \frac{5}{4}$

x^*_{12} is the midpoint of $[1,2]$

$x^*_{12} = 3/2$

$y^*_{12} = y^*_{11} = \frac{5}{4}$

$x^*_{21} = x^*_{11} = 1/2$

y^*_{21} is the midpoint of $[3/2, 2]$

$y^*_{21} = 7/4$

$x^*_{22} = x^*_{12} = 3/2$

$y^*_{22} = y^*_{21} = 7/4$

$f(1/2, 5/4) = 1/2 - 3\left(\frac{25}{16}\right)$

$f(3/2, 5/4) = 3/2 - 3\left(\frac{25}{16}\right)$

$f(1/2, 7/4) = 1/2 - 3\left(\frac{49}{16}\right)$

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$\Delta x = 1$

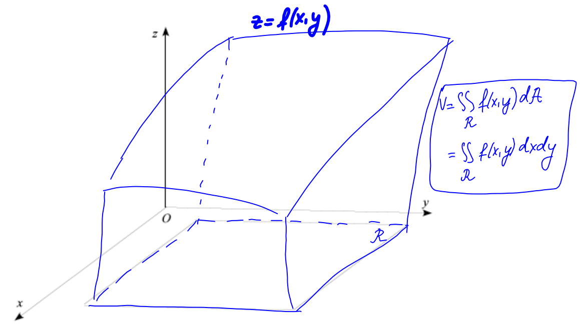
$\Delta y = 1/2$

$$\iint_R f(x,y)\,dA \approx f(1/2, 5/4)\cancelto{1/2}{\Delta x \Delta y} + f(3/2, 5/4)\cancelto{1/2}{\Delta x \Delta y} + f(1/2, 7/4)\cancelto{1/2}{\Delta x \Delta y} + f(3/2, 7/4)\cancelto{1/2}{\Delta x \Delta y}$$

$$= \boxed{1/2\left[\frac{1}{2} - 3\left(\frac{25}{16}\right) + \frac{3}{2} - 3\left(\frac{25}{16}\right) + 1/2 - 3\left(\frac{49}{16}\right) + 3/2 - 3\left(\frac{49}{16}\right)\right]}$$

Double integrals of positive functions can be interprepreted as volumes. Suppose that $f(x,y) \geq 0$ and f is defined on the rectangle $R = [a,b] \times [c,d]$. The graph of f is a surface with equation $z = f(a,b)$. Let S be the solid that lies above R and under the graph of f

$$S = \{(x,y,z) \in \mathbb{R}^3 | 0 \leq z \leq f(x,y), (x,y) \in R\}$$



$$V = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

If we partition R into subrectangles R_{ij} and choose (x_{ij}^*, y_{ij}^*) in R_{ij}, then we can approximate the part of S that lies above R_{ij} by a thin rectangular column with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. The volume of the column is

$$V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

If we follow this procedure for all rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S

$$V = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

Approximation becomes better if we use a finer partition P.

Theorem. If $f(x,y) \geq 0$ and f is continuous on the rectangle R, then the volume V of the solid that lie above R and under the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

Example 2. Evaluate the double integral

$$\iint_R (8-4y)\,dA$$

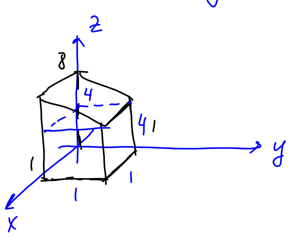
$$R = [0,1] \times [0,1]$$

by first identifying it as the volume of a solid.

$0 \leq x \leq 1$

$0 \leq y \leq 1$

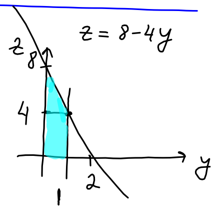
$z = 8-4y$



since $0 \leq y \leq 1$

$8 - 4(0) \geq 8-4y \geq 8-4(1)$

$8 \geq 8-4y \geq 4$



$$\iint_R (8-4y)\,dA = (1)(1)(4) + \frac{1}{2}(4)(1)(1) = \boxed{6}$$