Suppose f is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$.

We use notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called partial integration with respect to y.

integrate for y from c to d
treat x as a constant

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

The integral $\int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$ is called an iterated integral. Thus,

$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to x from a to b and then with respect to y from c to d.

Example 1. Evaluate the iterated integrals:

1. $\int_0^3 \int_0^1 \sqrt{x + y}\, dx dy$

$$\int_0^1 \sqrt{x+y}\, dx = \left. \frac{(x+y)^{3/2}}{3/2} \right|_{x=0}^{x=1} = \frac{2}{3}\left[(1+y)^{3/2} - y^{3/2}\right]$$

$$\int_0^3 \frac{2}{3}\left[(1+y)^{3/2} - y^{3/2}\right] dy = \frac{2}{3}\left[\frac{(1+y)^{5/2}}{5/2} - \frac{y^{5/2}}{5/2}\right]_{y=0}^{y=3} = \boxed{\frac{4}{15}\left[4^{5/2} - 3^{5/2} - 1\right]}$$

2. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2 + y^2 + 1}}\, dy dx$

$$\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\, dy$$

constant

$u = y^2 + (x^2+1)$
$du = 2y\,dy$
$y = 0 \to u = x^2+1$
$y = 1 \to u = x^2+2$

$$= \int_{x^2+1}^{x^2+2} \frac{x/2}{\sqrt{u}}\, du = \frac{x}{2} \int_{x^2+1}^{x^2+2} u^{-1/2}\, du = \frac{x}{2} \left. \frac{u^{1/2}}{1/2} \right|_{u=x^2+1}^{u=x^2+2} = x\left(\sqrt{x^2+2} - \sqrt{x^2+1}\right)$$

$$\int_0^1 x\left(\sqrt{x^2+2} - \sqrt{x^2+1}\right) dx = \frac{1}{2}\int_0^1 2x\sqrt{x^2+2}\, dx - \frac{1}{2}\int_0^1 2x\sqrt{x^2+1}\, dx$$

$u = x^2 + 2$
$du = 2x\,dx$
$x = 0 \to u = 2$
$x = 1 \to u = 3$

$u = x^2 + 1$
$du = 2x\,dx$
$x = 0 \to u = 1$
$x = 1 \to u = 2$

$$= \frac{1}{2}\int_2^3 \sqrt{u}\, du - \frac{1}{2}\int_1^2 \sqrt{u}\, du = \frac{1}{2} \left. \frac{u^{3/2}}{3/2} \right|_2^3 - \frac{1}{2} \left. \frac{u^{3/2}}{3/2} \right|_1^2 = \boxed{\frac{1}{3}\left[3^{3/2} - 2^{3/2}\right] - \frac{1}{3}\left[2^{3/2} - 1\right]}$$

Fubini's Theorem. If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\boxed{\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx}$$

Example 2. Calculate the double integral

$$\iint_R \left(xy^2 + \frac{y}{x}\right) dA = \int_2^3 \left[\int_{-1}^0 \left(xy^2 + \frac{y}{x}\right) dy\right] dx = \int_{-1}^0 \int_2^3 \left(xy^2 + \frac{y}{x}\right) dxdy$$

where $R = \{(x, y) | 2 \le x \le 3, -1 \le y \le 0\}$.

$$= \int_2^3 \left(x \frac{y^3}{3} + \frac{1}{x} \cdot \frac{y^2}{2}\right)\Big|_{y=-1}^{y=0} dx$$

$$= \int_2^3 \left[\frac{x}{3}\left(0 - (-1)^3\right) + \frac{1}{2x}\left(0 - (-1)^2\right)\right] dx$$

$$= \int_2^3 \left[+\frac{x}{3} - \frac{1}{2x}\right] dx$$

$$= \left[\frac{x^2}{6} - \frac{1}{2} \ln|x|\right]_2^3$$

$$= \frac{1}{6}(9-4) - \frac{1}{2}(\ln 3 - \ln 2)$$

$$= \boxed{\frac{5}{6} - \frac{1}{2} \ln \frac{3}{2}}$$

Example 3. Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.

$z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

$-1 \le x \le 1 \qquad -2 \le y \le 2$

$$V = \iint_R \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dA$$

$$= \int_{-1}^1 \int_{-2}^2 \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dy\, dx = \int_{-2}^2 \int_{-1}^1 \left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right) dx\, dy$$

$$= \int_{-1}^1 \left[\left(1 - \frac{x^2}{4}\right) y - \frac{y^3}{27}\right]_{y=-2}^{y=2} dx$$

$$= \int_{-1}^1 \left[\left(1 - \frac{x^2}{4}\right) \overset{(2-(-2))}{4} - \frac{16}{27}\right] dx$$

$$= \int_{-1}^1 \left[\left(4 - \frac{16}{27}\right) - x^2\right] dx$$

$$= \int_{-1}^1 \left[\frac{92}{27} - x^2\right] dx$$

$$= \left[\frac{92}{27} x - \frac{x^3}{3}\right]_{-1}^1$$

$$= \boxed{\frac{184}{27} - \frac{2}{3}}$$