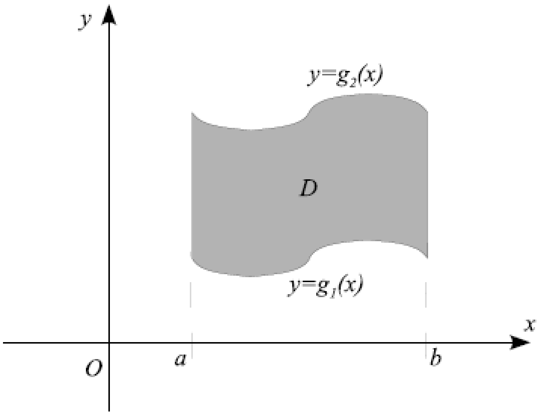
Chapter 13. **Multiple integrals.**
Section 13.3 **Double integrals over general regions.**

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of f, that is,

$$D = \{(x, y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$.



In order to evaluate $\iint\limits_D f(x,y)dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D and we let

$$F(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in D \\ 0, & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Then, by Fubini's Theorem,

$$\iint\limits_D f(x,y)dA = \iint\limits_R F(x,y)dA = \int_a^b \int_c^d F(x,y)\ dydx$$

$F(x, y) = 0$ if $y < g_1(x, y)$ or $y > g_2(x)$). Therefore, $\int\limits_c^d F(x, y)\ dy = \int\limits_c^{g_1(x)} F(x, y)\ dy + \int\limits_{g_1(x)}^{g_2(x)} F(x, y)\ dy + \int\limits_{g_2(x)}^{d} F(x, y)\ dy = \int\limits_{g_1(x)}^{g_2(x)} f(x, y)\ dy.$

If f is continuous on a type I region D such that

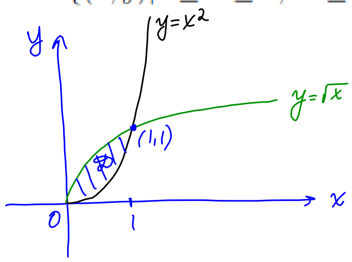
$$D = \{(x, y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then $$\iint\limits_D f(x,y)dA = \int\limits_a^b \int\limits_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$

Example 1. Evaluate the integral

$$\iint_D xy\, dA$$

if $D = \{(x,y) | 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}$.



$y = x^2$
$y = \sqrt{x}$
$x = 1$
$x = 0$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy \, dy\, dx$$

$$= \int_0^1 x \frac{y^2}{2} \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \frac{x}{2}\left((\sqrt{x})^2 - (x^2)^2 \right) dx$$

x $\quad$ x^4

$$= \frac{1}{2}\int_0^1 (x^2 - x^5)\, dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^6}{6} \right)\Big|_0^1$$

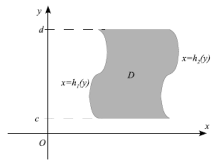
$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{6} \right)$$

$$= \boxed{\frac{1}{12}}$$

We also consider the plane region of **type II**, which can be expressed as

$$D = \{(x,y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous.



We can show that

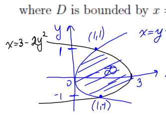
$$\boxed{\iint_D f(x,y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx dy}$$

where D is a type II region.

Example 2. Evaluate the double integrals

$$\iint_D (y^2 - x)\, dA,$$

where D is bounded by $x = y^2$ and $x = 3 - 2y^2$.



$y^2 \le x \le 3 - 2y^2$

Points of intersection:

$y^2 = 3 - 2y^2$
$3y^2 = 3$
$y = \pm 1$ $\quad$ $-1 \le y \le 1$

$$\int_{-1}^{1}\int_{y^2}^{3-2y^2} (y^2 - x)\, dx\, dy$$

$$= \int_{-1}^{1} \left(y^2 x - \frac{x^2}{2} \right)\Big|_{x=y^2}^{x=3-2y^2} dy$$

$$= \int_{-1}^{1} \left(y^2(3-2y^2) - \frac{(3-2y^2)^2}{2} - y^4 + \frac{y^4}{2} \right) dy$$

$$= \int_{-1}^{1} \left[3y^2 - 2y^4 - \frac{1}{2}(9 - 12y^2 + 4y^4) - \frac{y^4}{2} \right] dy$$

$$= \int_{-1}^{1} \left[3y^2 - 2y^4 - \frac{9}{2} + 6y^2 - 2y^4 - \frac{y^4}{2} \right] dy$$

$$= \int_{-1}^{1} \left(9y^2 - \frac{9}{2} y^4 - \frac{9}{2} \right) dy$$

$$= \left(\frac{9y^3}{3} - \frac{9}{2}\frac{y^5}{5} - \frac{9}{2} y \right)\Big|_{-1}^{1}$$

$$= \boxed{3(2) - \frac{9}{10}(2) - \frac{9}{2}(2)}$$

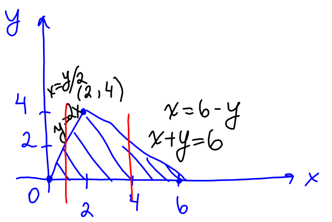
Properties of double integrals.
We assume that all of the following integrals exist. Then

1. $\iint\limits_D [f(x,y)+g(x,y)]dA = \iint\limits_D f(x,y)dA + \iint\limits_D g(x,y)dA$
2. $\iint\limits_D cf(x,y)dA = c\iint\limits_D f(x,y)dA$, where c is a constant
3. If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then
$$\iint\limits_D f(x,y)dA \geq \iint\limits_R g(x,y)dA$$
4. If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then
$$\iint\limits_D f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA$$
5. $\boxed{\iint\limits_D 1\, dA = A(D)}$
6. If $m \leq f(x,y) \leq M$ for all (x,y) in D, then
$$mA(D) \leq \iint\limits_D f(x,y)dA \leq MA(D).$$

Example 3. Evaluate
$$\iint\limits_D ye^x dA$$
if D is the triangular region with vertices $(0,0)$, $(2,4)$, and $(6,0)$.

$\int u\,dv = uv - \int v\,du$

$\int e^{ax}dx = \frac{1}{a}e^{ax} + C$



$\frac{y}{2} \leq x \leq 6-y$
$0 \leq y \leq 4$

$$= \int_0^4 \int_{y/2}^{6-y} ye^x\,dx\,dy$$
$$= \int_0^4 ye^x \Big|_{x=y/2}^{6-y} dy$$
$$= \int_0^4 y\left[e^{6-y} - e^{y/2}\right]dy$$
$$= \int_0^4 ye^{6-y}dy - \int_0^4 ye^{y/2}dy$$

$u = y \quad dv = e^{y/2}dy$
$du = dy \quad v = 2e^{y/2}$

$u = y \quad dv = e^{6-y}dy$
$du = dy \quad v = -e^{6-y}$

$$= -ye^{6-y}\Big|_0^4 + \int_0^4 (+e^{6-y})dy - \left[2ye^{y/2}\Big|_0^4 - \int_0^4 2e^{y/2}dy\right]$$
$$= -4e^2 - e^{6-y}\Big|_0^4 - 8e^2 + 4e^{y/2}\Big|_0^4$$
$$= -12e^2 - e^2 + e^6 + 4e^2 - 4$$
$$= \boxed{e^6 - 9e^2 - 4}$$

Example 4. Sketch the region of integration and change the order of integration for

$$\int_0^1 \int_{y^2}^{2-y} f(x,y)\,dxdy$$

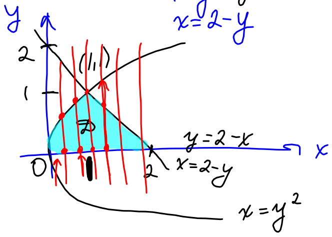
$\mathcal{D}$: $0 \le y \le 1$

$y^2 \le x \le 2-y$

$x = y^2,\ y = \sqrt{x}$

$x = 2-y$

$$\boxed{\int_0^1 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-x} f(x,y)\,dy\,dx}$$



① $0 \le x \le 1$

$0 \le y \le \sqrt{x}$

② $1 \le x \le 2$

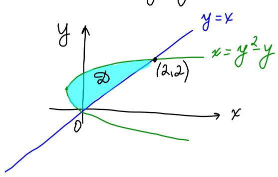
$0 \le y \le 2-x$

Example 5. Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$.

$$V = \iint_{\mathcal{D}} (3x^2+y^2)\,dA$$

$\mathcal{D}$: $y = x$

$x = y^2 - y$



Points of intersection:

$y = y^2 - y$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y_1 = 0$ $\quad$ $y_2 = 2$

$x_1 = y_1 = 0$ $\quad$ $x_2 = y_2 = 2$

$(0,0)$ $\quad$ $(2,2)$

Type II.

$y^2 - y \le x \le y$

$0 \le y \le 2$

$$V = \iint_{\mathcal{D}} (3x^2+y^2)\,dA = \int_0^2 \int_{y^2-y}^{y} (3x^2+y^2)\,dx\,dy$$

$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

$$= \int_0^2 \left(\frac{\cancel{3}x^3}{\cancel{3}} + y^2 x \right)\Big|_{x=y^2-y}^{x=y} dy$$

$$= \int_0^2 \left[y^3 + y^3 - (y^2-y)^3 - y^2(y^2-y) \right] dy$$

$$= \int_0^2 \left[2y^3 - (y^6 - \overset{3y^5}{3y^4 y} + \overset{3y^4}{3y^2 y^2} - y^3) - y^4 + y^3 \right] dy$$

$$= \int_0^2 \left[4y^3 - y^6 + 3y^5 - 4y^4 \right] dy$$

$$= \left(\frac{\cancel{4}y^4}{\cancel{4}} - \frac{y^7}{7} + \frac{\cancel{3}y^6}{\overset{2}{\cancel{6}}} - \frac{4y^5}{5} \right)\Big|_0^2$$

$$= \boxed{16 - \frac{128}{7} + \frac{64}{2} - \frac{4(32)}{5}}$$