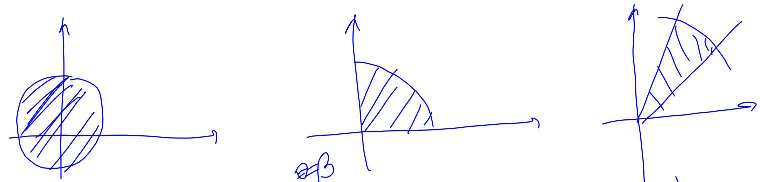
Section 13.5 **Double integrals in polar coordinates.**

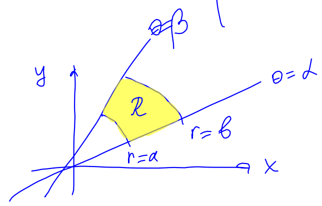
We want to evaluate

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

where R is a **polar rectangle**

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}$$





$(x,y) \mapsto (r,\theta)$

$x = r\cos\theta$

$y = r\sin\theta$

$$\boxed{dA = r\,dr\,d\theta}$$

We start with a partition of [a,b] into m subintervals

$$a = r_0 < r_1 < r_2 < ... < r_m = b$$

and a partition of $[\alpha, \beta]$ into n subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < ... < \theta_n = \beta$$

Then the circles $r = r_i$ and the rays $\theta = \theta_j$ determine a **polar partition** P of R into the small polar rectangles. The norm $\|P\|$ of the polar partition is the length of the longest diagonal of the polar subrectangles.

The "center" of the polar subrectangle

$$R_{ij} = \{(r,\theta) | r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_i + r_{i-1}) \qquad \theta_j^* = \frac{1}{2}(\theta_j + \theta_{j-1})$$

The area of the rectangle R_{ij} is

$$A(R_{ij}) = \Delta A_{ij} = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta_j = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j = r_i^* \Delta r_i \Delta\theta_j$$

where $\Delta\theta_j = \theta_j - \theta_{j-1}$, $\Delta r_i = r_i - r_{i-1}$.

The Cartesian coordinates of the center of R_{ij} are

$$(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$$

so a Riemann sum is

$$\sum_{i=1}^m \sum_{j=1}^n f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) \Delta A_{ij} = \sum_{i=1}^m \sum_{j=1}^n f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j$$

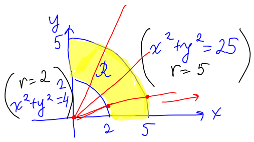
Change to polar coordinates in a double integral. If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \alpha - \beta \le 2\pi$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Example 1. Evaluate the integral

$$\iint_R xy\,dA$$

where R is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=25$.



Polar coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$0 \le \theta \le \frac{\pi}{2}$

$2 \le r \le 5$

$x^2+y^2=4$

$r^2\cos^2\theta + r^2\sin^2\theta = 4$

$r^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) = 4$

$r^2 = 4$

$r = 2$

$$\iint_R xy\,dA = \int_0^{\pi/2}\int_2^5 r\cos\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_2^5 r^3 \cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^4}{4}\Big]_{r=2}^{r=5} \cos\theta\sin\theta\,d\theta$$

$$= \int_0^{\pi/2}\underbrace{\left[\frac{625}{4} - \frac{16}{4}\right]}_{\frac{609}{4}} \cos\theta\sin\theta\,d\theta$$

$u = \sin\theta$
$du = \cos\theta\,d\theta$
$\theta = 0 \Rightarrow u = 0$
$\theta = \pi/2 \Rightarrow u = 1$

$$= \frac{609}{4}\int_0^1 u\,du$$

$$= \frac{609}{4}\,\frac{u^2}{2}\Big]_0^1$$

$$= \boxed{\frac{609}{8}}$$

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\},$$

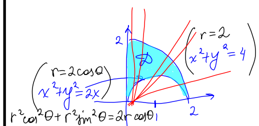
then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 2. Evaluate the integral

$$\iint_D x\,dA$$

where D is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$.



$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$x^2+y^2=2x$

$x^2-2x+y^2=0$

$(x-1)^2+y^2=1$

Polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$0 \le \theta \le \frac{\pi}{2}$

$2\cos\theta \le r \le 2$

$$\iint_D x\,dA = \int_0^{\pi/2}\int_{2\cos\theta}^2 r\cos\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_{2\cos\theta}^2 r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big]_{r=2\cos\theta}^{r=2} \cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{8}{3} - \frac{8}{3}\cos^3\theta\right)\cos\theta\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\cos\theta\,d\theta - \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\cos^4\theta = \left[\frac{1+\cos 2\theta}{2}\right]^2$

$= \frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$

$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)$

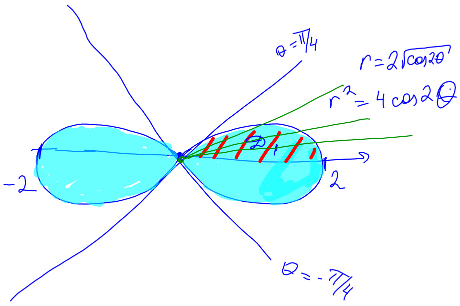
$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2\theta + \frac{\cos 4\theta}{2}\right)$

$$= \frac{8}{3}\sin\theta\Big|_0^{\pi/2} - \frac{8}{3}\cdot\frac{1}{4}\int_0^{\pi/2}\left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right]d\theta$$

$$= \frac{8}{3} - \frac{2}{3}\left[\frac{3}{2}\theta + \frac{2}{2}\sin 2\theta + \frac{1}{2}\,\frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \boxed{\frac{8}{3} - \frac{\pi}{2}}$$

Example 3. Use a double integral to find the area of the region enclosed by the lemniscate $r^2 = 4\cos 2\theta$.



$$A = 4\iint_{D_1} dA$$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$0 \le \theta \le \pi/4$

$0 \le r \le 2\sqrt{\cos 2\theta}$

$$A = 4\int_0^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \frac{r^2}{2}\Big]_{r=0}^{r=2\sqrt{\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} \frac{4}{2}\cos 2\theta\,d\theta$$

$$= 8\int_0^{\pi/4} \cos 2\theta\,d\theta$$

$$= 4\sin 2\theta\Big|_0^{\pi/4}$$

$$= 4\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \boxed{4}$$

Example 4. Use a double integral to find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

$r(r) = 3\cos\theta(r)$

$r^2 = 3r\cos\theta$

$x^2 + y^2 = 3x$

$(x - 3/2)^2 + y^2 = 9/4$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$r = 1 + \cos\theta$

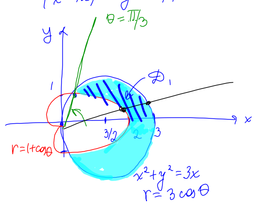
$\theta = 0,\ r = 2$

$\theta = \pi/2,\ r = 1$

$\theta = \pi,\ r = 0$

$\theta = \frac{3\pi}{2},\ r = 1$

$\theta = 2\pi,\ r = 2.$



$$A = 2\iint_{D_1} dA$$

point of intersection:

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = 1/2$

$\theta = \pi/3$

$0 \le \theta \le \pi/3$

$1 + \cos\theta \le r \le 3\cos\theta$

$$A = 2\int_0^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/3} \frac{r^2}{2}\Big]_{r=1+\cos\theta}^{r=3\cos\theta} d\theta$$

$$= \int_0^{\pi/3}\left[9\cos^2\theta - (1+\cos\theta)^2\right] d\theta$$

$$= \int_0^{\pi/3}\left[9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right] d\theta$$

$$= \int_0^{\pi/3}\left[\overset{4(1+\cos 2\theta)}{\cancel{8\cos^2\theta}} - 1 - 2\cos\theta\right] d\theta$$

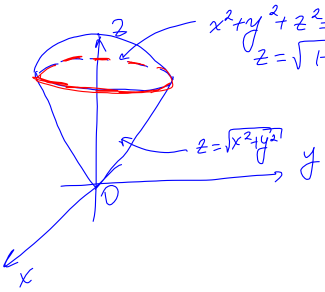
$$= \int_0^{\pi/3}\left[3 + 4\cos 2\theta - 2\cos\theta\right] d\theta$$

$$= \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_{\theta=0}^{\theta=\pi/3}$$

$$= \left[3\frac{\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right]$$

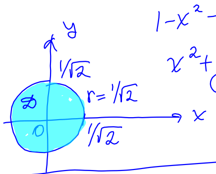
$$= \boxed{\pi}$$

Example 5. Use polar coordinates to find the volume above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.



intersection:

$$\sqrt{1-x^2-y^2} = \sqrt{x^2+y^2}$$
$$1-x^2-y^2 = x^2+y^2$$
$$x^2+y^2 = 1/2$$
$$r = 1/\sqrt{2}$$



Polar coordinates:

$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

$0 \le \theta \le 2\pi$
$0 \le r \le 1/\sqrt{2}$

$$V = \iint_{\mathcal{D}} \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA$$
$$= \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta$$
$$= -\frac{1}{2}\int_0^{2\pi}\int_0^{1/\sqrt{2}} -2r\sqrt{1-r^2}\,dr\,d\theta - \int_0^{2\pi}\int_0^{1/\sqrt{2}} r^2\,dr\,d\theta$$

$u = 1-r^2$
$du = -2r\,dr$
$r=0 \to u=1$
$r=1/\sqrt{2} \to u=1/2$

$\frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right)$

$$= -\frac{1}{2}\int_0^{2\pi}\int_1^{1/2} \sqrt{u}\,du\,d\theta - \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{1/\sqrt{2}} d\theta$$
$$= -\frac{1}{2}\int_0^{2\pi} \left[\frac{2u^{3/2}}{3}\right]_1^{1/2} d\theta - \frac{1}{6\sqrt{2}}\int_0^{2\pi} d\theta$$
$$= -\frac{1}{2}\cdot\frac{2}{3}\int_0^{2\pi} \left[\left(\frac{1}{2}\right)^{3/2} - 1\right] d\theta - \frac{1}{6\sqrt{2}}(2\pi)$$
$$= -\frac{1}{3}\left[\frac{1}{2\sqrt{2}} - 1\right]\int_0^{2\pi} d\theta - \frac{2\pi}{6\sqrt{2}}$$
$$= \boxed{2\pi\left[-\frac{1}{6\sqrt{2}} + \frac{1}{3} - \frac{1}{6\sqrt{2}}\right]}$$