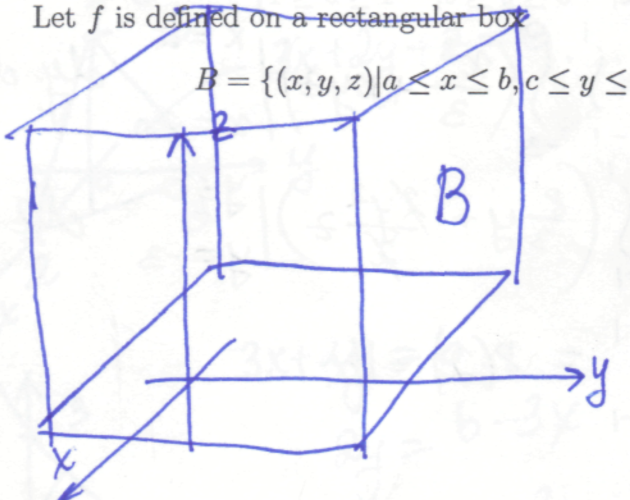
Chapter 13. Multiple integrals.
Section 13.8 Triple integrals.

We want to define the triple integrals for functions of three variables.
Let f is defined on a rectangular box

$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\} = [a, b] \times [c, d] \times [r, s]$$



We partition the intervals $[a, b]$, $[c, d]$, and $[r, s]$ as follows:

$$a = x_0 < x_1 < ... < x_m = l$$

$$c = y_0 < y_1 < ... < y_n = m$$

$$r = z_0 < z_1 < ... < z_k = n$$

The planes through these partition points parallel to coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

The volume of B_{ijk} is

$$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$$

where $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and $\Delta z_k = z_k - z_{k-1}$.

Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \in B_{ijk}$. We define the **norm** $\|P\|$ of the partition P to be the length of the longest diagonal of all the boxes B_{ijk}.

Definition. The **triple integral** of f over the box B is

$$\iiint\limits_B f(x, y, z) dV = \lim_{\|P\| \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk}$$

if this limit exists.

Fubini's Theorem for triple integrals. If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint\limits_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

There are five other possible orders in which we can integrate.

Example 1. Evaluate the integral $\iiint_E (x^2+yz)dV$, where

$$E=\{(x,y,z)|0\le x\le 2,\ -3\le y\le 0,\ -1\le z\le 1\}$$

$$I=\int_{-1}^{1}\int_{-3}^{0}\int_{0}^{2}(x^2+yz)\,dx\,dy\,dz=\int_{-1}^{1}\int_{-3}^{0}\left(\frac{x^3}{3}+xyz\right)\Big|_{x=0}^{x=2}dy\,dz$$

$$=\int_{-1}^{1}\int_{-3}^{0}\left(\frac{8}{3}+2yz\right)dy\,dz=\int_{-1}^{1}\left(\frac{8}{3}y+\frac{\cancel{2}y^2}{\cancel{2}}z\right)\Big|_{y=-3}^{y=0}dz$$

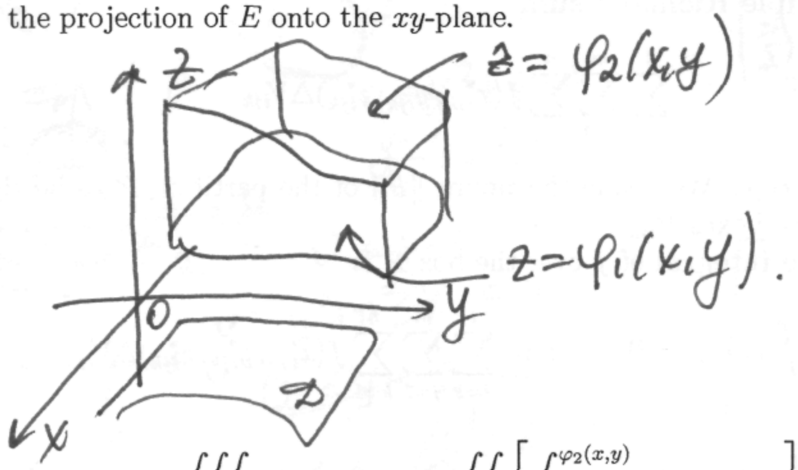
$$=\int_{-1}^{1}(8-9z)\,dz=\left(8z-\frac{9z^2}{2}\right)\Big|_{-1}^{1}=8(2)=16.$$

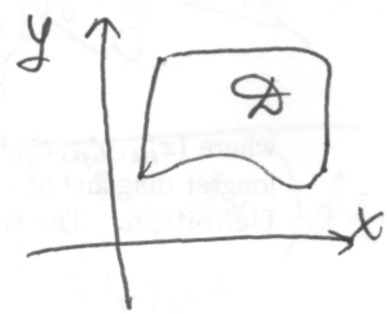
Now we define the **triple integral over a general bounded region** E in three-dimensional space.

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E=\{(x,y,z)|(x,y)\in D,\varphi_1(x,y)\le z\le\varphi_2(x,y)\}$$

where D is the projection of E onto the xy-plane.

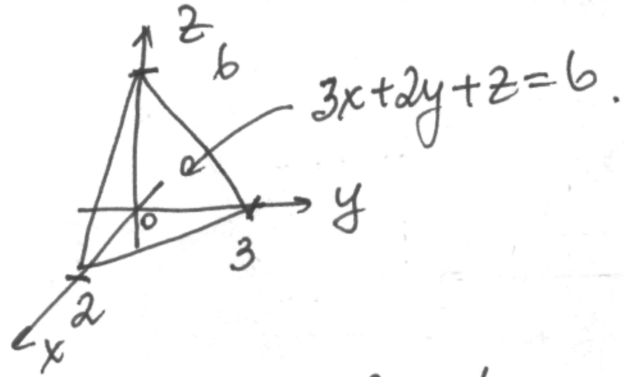


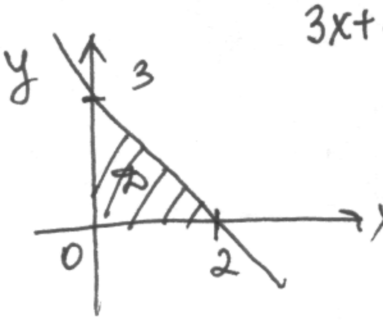


Then

$$\iiint_E f(x,y,z)dV=\iint_D\left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)}f(x,y,z)dz\right]dA$$

Example 2. Evaluate $\iiint\limits_E x\,dV$, where E is bounded by planes $x = 0$, $y = 0$, $z = 0$, and $3x + 2y + z = 6$.





$3x+2y=6.$

$2y = 6-3x$

$y = 3-\frac{3}{2}x$

$$\iiint\limits_E x\,dV$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} x\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} x z \Big|_{z=0}^{z=6-3x-2y} dy\,dx$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} \left[6x-3x^2-2xy\right] dy\,dx$$

$$= \int_0^2 \left[6xy-3x^2y-2x\frac{y^2}{2}\right]_{y=0}^{y=3-\frac{3}{2}x} dx$$

$$= \int_0^2 \left[6x\left(3-\frac{3}{2}x\right) - 3x^2\left(3-\frac{3}{2}x\right) - x\left(3-\frac{3}{2}x\right)^2\right] dx$$

$$= \int_0^2 \left[18x-9x^2-9x^2+\frac{9}{2}x-9x\left(1-x+\frac{1}{4}x^2\right)\right] dx$$

$$= \int_0^2 \left[18x-18x^2+\frac{9}{2}x-9x+9x^2-\frac{9}{4}x^3\right] dx$$

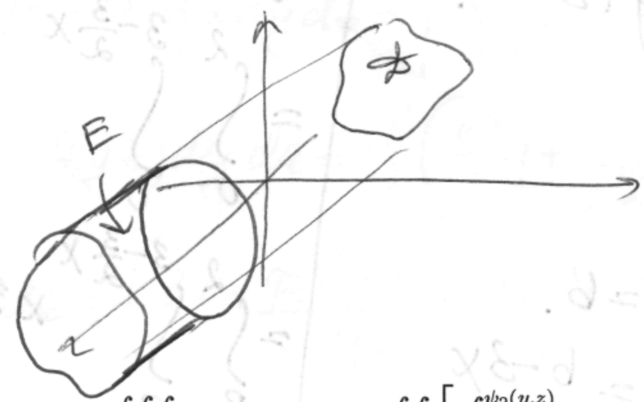
$$= \int_0^2 \left[\frac{27}{2}x-9x^2-\frac{9}{4}x^3\right] dx = \left[\frac{27}{4}x^2-\frac{9x^3}{3}-\frac{9}{4}\frac{x^3}{3}\right]_0^2$$

$$= \left[\frac{27}{4}4-3(8)-\frac{3}{4}(8)\right] = 27-24-6 = \boxed{-3}.$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) | (y, z) \in D, \psi_1(y, z) \leq x \leq \psi_2(y, z)\}$$

where D is the projection of E onto the yz-plane.



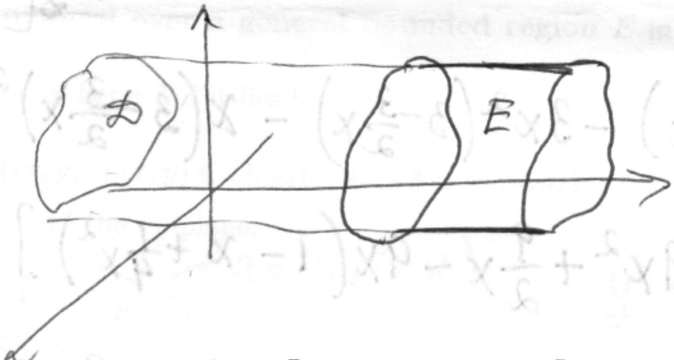
Then

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{\psi_1(y,z)}^{\psi_2(y,z)} f(x, y, z) dx \right] dA$$

A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) | (x, z) \in D, \chi_1(x, z) \leq x \leq \chi_2(x, z)\}$$

where D is the projection of E onto the xz-plane.



Then

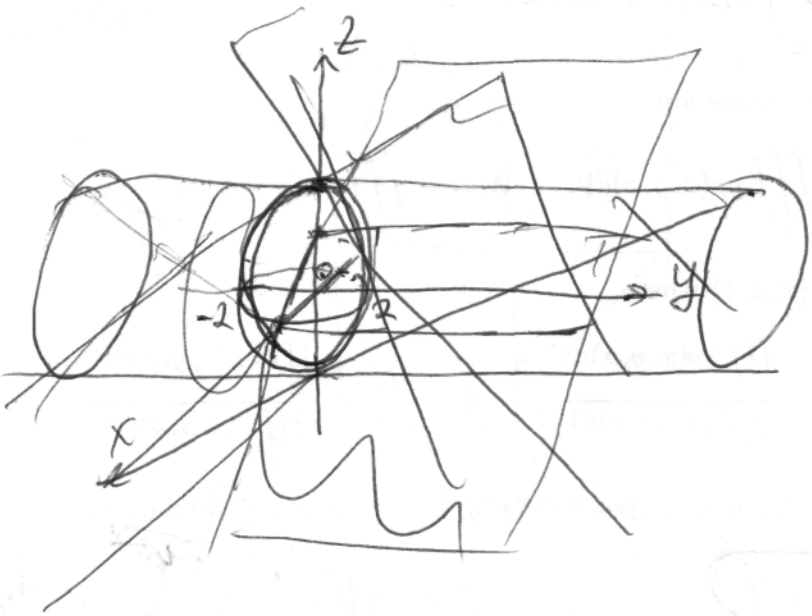
$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{\chi_1(x,z)}^{\chi_2(x,z)} f(x, y, z) dy \right] dA$$

Applications of triple integrals.

$$V(E) = \iiint_E dV$$

Example 3. Find the volume of the solid bounded by the elliptic cylinder $4x^2 + z^2 = 4$ and the planes $y = 0$ and ~~$x + z = 1$.~~ $y = z + 2$.

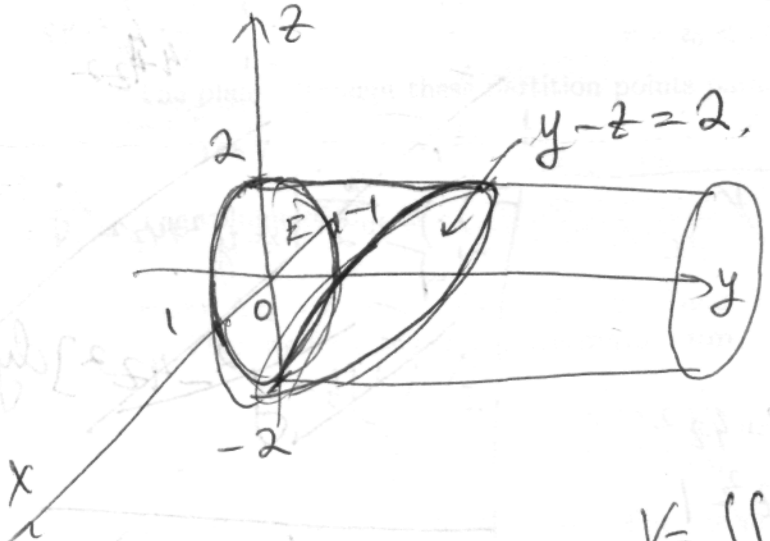
xz plane. $x+z=1$, $y-z=2$



$4x^2+z^2=4$.

~~$y=1$~~ $y-z=2$.

$y=0 \to z=-2$



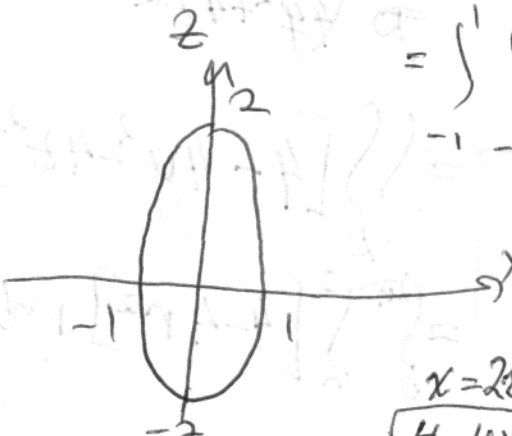
$0 \le y \le z+2$.

$-1 \le x \le 1$

$-\sqrt{4-4x^2} \le z \le \sqrt{4-4x^2}$.

$$V = \iiint_E dV = \int_{-1}^{1}\int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}}\int_0^{z+2} dy\,dz\,dx$$

(xz)-plane



$$= \int_{-1}^{1}\int_{-\sqrt{}}^{\sqrt{}} (z+2)\,dz\,dx = \int_{-1}^{1}\left(\frac{z^2}{2}+2z\right)\Bigg|_{-\sqrt{}}^{\sqrt{}} dx =$$

$$= \int_{-1}^{1}\left(\cancel{(4-4x^2)} + 4\sqrt{4-4x^2}\right)dx = 4\int_{-1}^{1}\sqrt{4-4x^2}\,dx$$

$x = 2\sin t$

$\sqrt{4-4x^2} = 2\cos t$

$dx = 2\cos t\,dt$

$$= 4\int_{-\pi/2}^{\pi/2} 2\cos t\, 2\cos t\,dt = \frac{16}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2t)\,dt = 8\pi$$

If the density function of a solid object that occupies the region E is $\rho(x, y, z)$ in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

$$m = \iiint_E \rho(x, y, z)\,dV$$

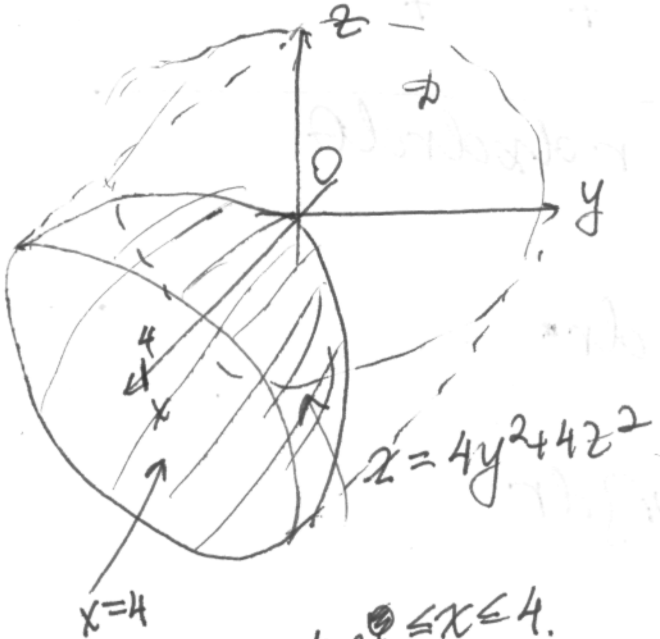
and its **moments** about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x, y, z)\,dV, \qquad M_{xz} = \iiint_E y\rho(x, y, z)\,dV, \qquad M_{xy} = \iiint_E z\rho(x, y, z)\,dV$$

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m} = \frac{\iiint_E x\rho(x, y, z)\,dV}{\iiint_E \rho(x, y, z)\,dV}, \qquad \bar{y} = \frac{M_{xz}}{m} = \frac{\iiint_E y\rho(x, y, z)\,dV}{\iiint_E \rho(x, y, z)\,dV}, \qquad \bar{z} = \frac{M_{xy}}{m} = \frac{\iiint_E z\rho(x, y, z)\,dV}{\iiint_E \rho(x, y, z)\,dV}$$

Example 4. Find the center of mass of a solid E with density $\rho(x, y, z) = x$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.



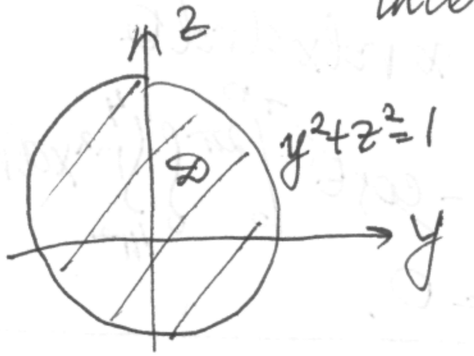
$x = 4y^2 + 4z^2$

$$\bar{x} = \frac{\iiint_E x\rho\,dV}{\iiint_E \rho\,dV}; \quad \bar{y} = \frac{\iiint_E y\rho\,dV}{\iiint_E \rho\,dV}$$

$4y^2 + 4z^2 \le x \le 4.$

intersection: $4y^2 + 4z^2 = 4$

$y^2 + z^2 = 1$



$y^2 + z^2 = 1$

$$\iiint_E \rho\,dV = \iint_D \left(\int_{4y^2+4z^2}^{4} x\,dx \right) dA$$

$$= \iint_D \left[8 - \frac{(4y^2+4z^2)^2}{2} \right] dA$$

$$= \iint_D \left[8 - 8(y^2+z^2)^2 \right] dA$$

Polar coordinates:

$y = r\cos\theta$; $dA = r\,dr\,d\theta$

$z = r\sin\theta$;

$0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_0^1 [8 - 8r^4]\, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 [8r - 8r^5]\,dr =$$

$$= 2\pi \int_0^{4} (8)\left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1$$

$$= 16\pi\left[\frac{1}{2} - \frac{1}{6}\right] = \frac{16\pi}{3}$$

$$\iiint_E x^2\,dV = \cancel{\iiint}_{D}\, x^2 = \int_0^{2\pi}\int_0^1\int_{4r^2}^{4} r x^2\,dx\,\cancel{r}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\,\frac{x^3}{3}\Big]_{4r^2}^{4}\,dr\,d\theta = \frac{2\pi}{3}\int_0^1 \left[64 - 64r^6\right] dr$$

$$= \frac{128\pi}{3}\left[r - \frac{r^7}{7}\right]_0^1 = \frac{128\pi}{3}\cdot\frac{6}{7} = \frac{256\pi}{7}$$

$$\iiint_E xy\,dV = \int_0^{2\pi}\int_0^1\int_{4r^2}^{4} r\cos\theta\; x\; r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\cos\theta\,d\theta \cdot \int_0^1 r^2\,\frac{x^2}{2}\Big]_{4r^2}^{4}\,dr$$

$$= \cancel{\sin\theta}\Big|_0^{2\pi} \cdot \int_0^1 \frac{r^2}{2}\left[16 - 16r^4\right] dr$$

$$= 0.$$

$$\iiint_E xz\,dV = \int_0^{2\pi}\int_0^1\int_{4r^2}^{4} r\sin\theta\; x\; r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\sin\theta\,d\theta \cdot \int_0^1\int_{4r^2}^{4} r^2 x\,dr\,d\theta = -\cancel{\cos\theta}\Big|_0^{2\pi}\int_0^1\int_{4r^2}^{4} r^2 x\,dr\,d\theta$$

$$= 0.$$

$$\bar{x} = \frac{\frac{256\pi}{7}}{\frac{16\pi}{3}} = \frac{256\cdot 3}{7\cdot 16} = \frac{48}{7},\quad \bar{y} = \bar{z} = 0.$$

$$\boxed{\left(\frac{48}{7}, 0, 0\right)}$$