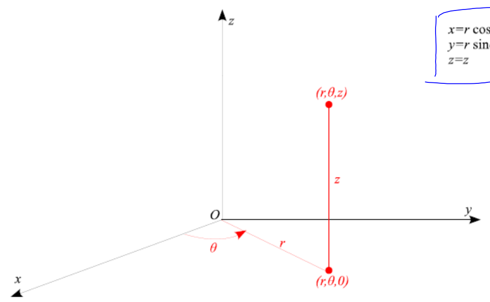
## Section 13.9 Cylindrical and spherical coordinates.

**Cylindrical coordinate system:**

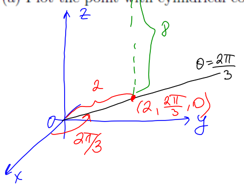



$$\boxed{\begin{aligned} x&=r\cos\theta\\ y&=r\sin\theta\\ z&=z\end{aligned}}$$

To convert from rectangular to cylindrical coordinates we use

$(2, \frac{2\pi}{3}, 8)$ $\boxed{r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad z = z}$

**Example 1.**
(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 8)$ and find its rectangular coordinates.




$r=2$
$\theta=\frac{2\pi}{3}$
$z=8.$

$x = r\cos\theta = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$
$y = r\sin\theta = 2\sin\frac{2\pi}{3} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$
$z = z = 8$

$\boxed{(-1, \sqrt{3}, 8)}$ rectangular coordinates.

(b) Find the cylindrical coordinates of the point with rectangular coordinates $(-\sqrt{2}, \sqrt{2}, 0)$.

$x = -\sqrt{2} < 0$
$y = \sqrt{2} > 0$
$z = 0$

$r^2 = x^2 + y^2 = 2 + 2 = 4, \quad r = 2$
$\tan\theta = \frac{y}{x} = \frac{\sqrt{2}}{-\sqrt{2}} = -1, \quad \theta = \frac{3\pi}{4}$
$z = 0$

cylindrical coordinates are $\boxed{(2, \frac{3\pi}{4}, 0)}$

**Example 2.** Sketch the solid given by the inequalities

first octant

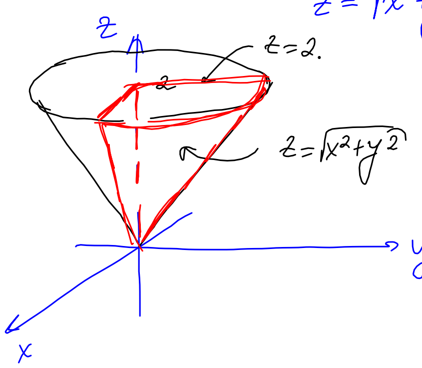
$$0 \le \theta \le \pi/2, \qquad r \le z \le 2$$

$z = r$, $z = 2$ – plane

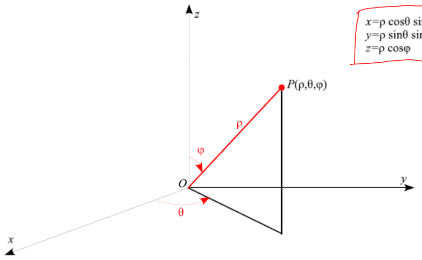
write the equation of the surface in cartesian coordinates.

$r^2 = x^2 + y^2, \quad r = \sqrt{x^2+y^2}$

$z = \sqrt{x^2+y^2}$ cone (upper half).



**Spherical coordinate system:**




$x = \rho \cos\theta \sin\varphi$
$y = \rho \sin\theta \sin\varphi$
$z = \rho \cos\varphi$

To convert from rectangular to ~~cylindrical~~ spherical coordinates we use

$$\boxed{\rho^2 = x^2 + y^2 + z^2 \quad \cos\varphi = \frac{z}{\rho} \quad \cos\theta = \frac{x}{\rho\sin\varphi}}$$



**Example 3.**

1. The point $(1, \pi/4, \pi/6)$ is given in spherical coordinates. Find its rectangular coordinates.

$\rho = 1$, $\theta = \pi/4$, $\varphi = \pi/6$

$x = \rho\cos\theta\sin\varphi = 1\cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{4}$

$y = \rho\sin\theta\sin\varphi = 1\sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{4}$

$z = \rho\cos\varphi = 1\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

rectangular coordinates $\boxed{\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right)}$

2. The point $(-\sqrt{3}, -3, -2)$ is given in rectangular coordinates. Find its spherical coordinates.

$\rho^2 = x^2+y^2+z^2 = 3+9+4 = 16$, $\boxed{\rho = 4}$

$\cos\varphi = \frac{z}{\rho} = -\frac{2}{4} = -\frac{1}{2}$, $\boxed{\varphi = \frac{2\pi}{3}}$

$\cos\theta = \frac{x}{\rho\sin\varphi} = \frac{-\sqrt{3}}{4\sin\frac{2\pi}{3}} = \frac{-\sqrt{3}}{4\sqrt{3}/2} = -\frac{1}{2}$; $\theta = \pi + \frac{\pi}{3} = \boxed{\frac{4\pi}{3}}$

spherical coordinates: $\boxed{\left(4, \frac{4\pi}{3}, \frac{2\pi}{3}\right)}$

To convert from spherical to cylindrical coordinates we use

$$\boxed{\theta = \theta \quad z = \rho\cos\varphi \quad r = \sqrt{\rho^2 - z^2}}$$

To convert from cylindrical to spherical coordinates we use

$$\boxed{\theta = \theta \quad \rho = \sqrt{r^2 + z^2} \quad \cos\varphi = \frac{z}{\rho}}$$

**Example 4.**

1. The point $(8, \pi/6, \pi/2)$ is given in spherical coordinates. Find its cylindrical coordinates.

$\rho = 8$, $\theta = \pi/6$, $\varphi = \pi/2$

Cylindrical coordinates: $\boxed{(8, \pi/6, 0)}$

$\theta = \pi/6$

$z = \rho\cos\varphi = 8\cos\frac{\pi}{2} = 0$

$r = \sqrt{\rho^2 - z^2} = \sqrt{64 - 0} = 8$



2. The point $(\sqrt{2}, \pi/4, 0)$ is given in cylindrical coordinates. Find its spherical coordinates.

$r = \sqrt{2}$, $\theta = \pi/4$, $z = 0$

Spherical coordinates: $\boxed{(\sqrt{2}, \frac{\pi}{4}, \pi/2)}$

$\theta = \pi/4$

$\rho = \sqrt{r^2 + z^2} = \sqrt{2 + 0} = \sqrt{2}$

$\cos\varphi = \frac{z}{\rho} = \frac{0}{\sqrt{2}} = 0$, $\varphi = \pi/2$

**Example 5.** Sketch the solid described by the inequalities

$$-\pi/2 \le \theta \le \pi/2, \quad 0 \le \varphi \le \pi/6, \quad 0 \le \rho \le \sec\varphi$$

$x = \rho\cos\theta\sin\varphi$
$y = \rho\sin\theta\sin\varphi$
$z = \rho\cos\varphi$

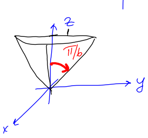
$\rho = \sec\varphi = \frac{1}{\cos\varphi}$ $\qquad \rho = \frac{1}{\cos\varphi}$

$\rho\cos\varphi = 1$ − plane

$z = 1$

$\varphi = \pi/6$ − cone

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



**Example 6.** Write the equation

$$x^2 - y^2 - 2z^2 = 4$$

in cylindrical and spherical coordinates.

cylindrical:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

spherical:

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

use: $\cos 2\theta = \cos^2 - \sin^2\theta$.

$\boxed{r^2\cos 2\theta - 2z^2 = 4}$

$\boxed{\rho^2(\cos 2\theta\ \sin^2\varphi - 2\cos^2\varphi) = 4}$