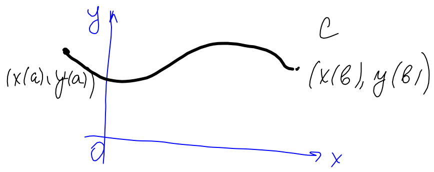
Section 14.2 **Line integrals.**

$\int_C f(x,y)\,ds$

Let C be a smooth plane curve with parametric equations

$$C: \quad x = x(t), \qquad y = y(t), \qquad a \le t \le b$$



A partition of the parameter interval $[a,b]$ by points t_i with

$$a = t_0 < t_1 < \ldots < t_n = b$$

determine a partition P of the curve by points $P_i(x_i, y_i)$, where $x_i = x(t_i)$, $y_i = y(t_i)$, $z_i = z(t_i)$. Points P_i divide C into n subarcs with length Δs_1, Δs_2,...,Δs_n. The **norm** $\|P\|$ of the partition is the longest of these lengths. We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc.

Definition. If f is defined on a smooth curve C given by

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

then the line integral of f along C with respect to arc length is

$$\int_C f(x,y)\,ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Since

$$ds = \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt$$

then

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\,dt}$$

The value of the line integral does not depend on the parametrization of the curve provided that the curve is traversed exactly once as t increases from a to b.

Example 1. Evaluate the line integral $\int_C x\,ds$, where C is a given by $x = t^3$, $y = t$, $0 \le t \le 1$.

$x'(t) = 3t^2, \quad y'(t) = 1$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$
$$= \sqrt{9t^4+1}\,dt$$

$$\int_C x\,ds = \frac{1}{36}\int_0^1 36t^3\sqrt{9t^4+1}\,dt \qquad \begin{array}{l} u = 9t^4+1 \\ du = 36t^3\,dt \\ 0 \to 1 \\ 1 \to 10 \end{array}$$

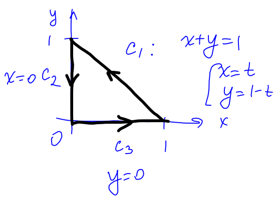
$$= \frac{1}{36}\int_1^{10} \sqrt{u}\,du = \frac{1}{36}\,\frac{2u^{3/2}}{3}\Big]_1^{10} = \boxed{\frac{1}{54}\left(10^{3/2} - 1\right)}$$

Suppose now that C is a piecewise-smooth curve; that is, C is a union of a finite number of smooth ~~functions~~ curves C_1, C_2,...,C_n. Then

$$\boxed{\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds}$$

Example 2. Evaluate $\int_C (x+y)\,ds$ if C consists of line segments from (1,0) to (0,1), from (0,1) to (0,0), and from (0,0) to ~~(0,1)~~ (1,0).

$ds = \sqrt{[x']^2 + [y']^2}\,dt$



C_1: $x = t$, $y = 1-t$. t changes from 1 to 0
$ds = \sqrt{1+1}\,dt = \sqrt{2}\,dt$

C_2: $x = 0$, $y = t$, t from 1 to 0
$ds = \sqrt{0+1}\,dt = dt$

C_3: $x = t$, $y = 0$, t from 0 to 1
$ds = \sqrt{0+1}\,dt = dt$

$$\int_C (x+y)\,ds = \int_{C_1}(x+y)\,ds + \int_{C_2}(x+y)\,ds + \int_{C_3}(x+y)\,ds$$

$$= \int_1^0 (t + (1-t))\sqrt{2}\,dt + \int_1^0 (0+t)\,dt + \int_0^1 (t+0)\,dt$$

$$= \int_1^0 \sqrt{2}\,dt + \int_1^0 t\,dt + \int_0^1 t\,dt = \boxed{-\sqrt{2}}$$

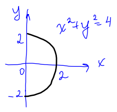
Physical interpretation of a line integral $\int_C f(x,y)ds$. Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the mass of wire is

$$\boxed{m = \int_C \rho(x,y)ds}$$

The center of mass of the wire with density function ρ is at the point $(\bar{x},\bar{y})$, where

$$\boxed{\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds \quad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds}$$

Example 3. A thin wire is bent into the shape of a semicircle $x^2+y^2=4$, $x \geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.



$m = \int_C \rho(x,y)ds$

$\rho(x,y) = k$

$x = 2\cos t \qquad x'(t) = -2\sin t$

$y = 2\sin t \qquad y'(t) = 2\cos t$

$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$ds = \sqrt{[x']^2+[y']^2}\,dt = \sqrt{4\sin^2 t + 4\cos^2 t} = 2dt$

$m = \int_{-\pi/2}^{\pi/2} k \cdot 2dt = \boxed{2k\pi}$

<u>Center of mass</u>.

$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds, \quad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds$

$= 0$ (by symmetry)

$ds = 2dt$

$\bar{x} = \frac{1}{\cancel{2k}\pi}\int_{-\pi/2}^{\pi/2} \overbrace{\cancel{2}\cos t}^{x}\ \overbrace{\cancel{k}}^{\rho}\ \overbrace{2dt}^{ds}$

$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos t\, dt$

$= \frac{2}{\pi} \sin t \Big]_{-\pi/2}^{\pi/2} = \boxed{\frac{4}{\pi}}$

center of mass @ $\boxed{\left(\frac{4}{\pi}, 0\right)}$

Line integrals of f along C with respect to x and y:

$$\int_C f(x,y)dx = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta x_i, \quad \Delta x_i = x_i - x_{i-1}$$

$$\int_C f(x,y)dy = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta y_i, \quad \Delta y_i = y_i - y_{i-1}$$

If $x = x(t)$, $y = y(t)$, then $dx = x'(t)dt$, $dy = y'(t)dt$, and

$$\boxed{\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt}$$

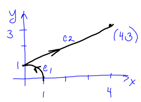
$$\boxed{\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt}$$

In general, we will write

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C P(x,y)dx + \int_C Q(x,y)dy$$

$= \int_a^b [P(x(t),y(t))\,x'(t) + Q(x(t),y(t))y'(t)]dt$

Example 3. Evaluate $\int_C x\sqrt{y}dx + 2y\sqrt{x}dy$, if C consists of the arc of the circle $x^2+y^2=1$ from (1,0) to (0,1) and the line segment from (0,1) to (4,3).



C_1: $x = \cos t \qquad dx = -\sin t\,dt$

$y = \sin t \qquad dy = \cos t\,dt$

$0 \leq t \leq \pi/2$

C_2: $\frac{x-0}{4-0} = \frac{y-1}{3-1}$

$\frac{x}{4} = \frac{y-1}{2}$

$x = 2y-2$

$x = 2t-2 \qquad dx = 2dt$

$y = t \qquad dy = dt$

$1 \leq t \leq 3$

$\int_C x\sqrt{y}\,dx + 2y\sqrt{x}\,dy = \int_{C_1} x\sqrt{y}\,dx + 2y\sqrt{x}\,dy + \int_{C_2} x\sqrt{y}\,dx + 2y\sqrt{x}\,dy$

$= \int_0^{\pi/2} [\cos t\sqrt{\sin t}\,(-\sin t) + 2\sin t\sqrt{\cos t}\,\cos t]dt$

$+ \int_1^3 [(2t-2)\sqrt{t}\,2 + 2t\sqrt{2t-2}]dt$

$u = \sin t$, $du = \cos t\,dt$ $\Big|$ $0 \to 0$, $\pi/2 \to 1$

$u = \cos t$, $du = -\sin t\,dt$ $\Big|$ $0 \to 1$, $\pi/2 \to 0$

$= -\int_0^{\pi/2} \cos t(\sin t)^{3/2}dt + 2\int_0^{\pi/2} \sin t(\cos t)^{3/2}dt$

$+ 4\int_1^3 t^{3/2}dt - 4\int_1^3 t^{1/2}dt + 2\int_1^3 t\sqrt{2t-2}\,dt$

$u = 2t-2$, $du = 2dt$, $t = \frac{u+2}{2}$ $\Big|$ $1 \to 0$, $3 \to 4$

$= -\int_0^1 u^{3/2}du + 2\int_{1\to 0}^{0\to 1} u^{3/2}du + 4\cdot\frac{2}{5}t^{5/2}\Big]_1^3 - 4\cdot\frac{2}{3}t^{3/2}\Big]_1^3$

$+ \int_0^4 \frac{u+2}{2}\sqrt{u}\,du$

$= \int_0^1 u^{3/2}du + \frac{8}{5}[3^{5/2}-1] - \frac{8}{3}[3^{3/2}-1] + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{5}u^{5/2}\Big|_0^4 + \cancel{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^4\right)$

$= \frac{2}{5}u^{5/2}\Big]_0^1 + \frac{8}{5}3^{5/2} - \frac{8}{5} - \frac{8}{3}3^{3/2} + \frac{8}{3} + \frac{32}{5} + \frac{16}{3}$

$= \boxed{\frac{2}{5} + \frac{8}{5}3^{5/2} - \frac{8}{5} - \frac{8}{3}3^{3/2} + \frac{8}{3} + \frac{32}{5} + \frac{16}{3}}$

A given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an orientation of a curve C, with the positive direction corresponding to increasing value of the parameter t.

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, then we have

$$\boxed{\int_{-C} f(x,y)dx = -\int_C f(x,y)dx} \quad \boxed{\int_{-C} f(x,y)dy = -\int_C f(x,y)dy}$$

but

$$\boxed{\int_{-C} f(x,y)ds = \int_C f(x,y)ds}$$

Line integrals in space.

Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t), \quad y = y(t) \quad z = z(t), \quad a \le t \le b$$

or by a vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$. We define the linear integral of f along C with respect to arc length as

$$\boxed{\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}dt = \int_a^b f(\vec{r}(t))|\vec{r}'(t)|dt}$$

If $f(x,y,z) = 1$, then

$$\boxed{\int_c ds = \int_a^b |\vec{r}'(t)dt| = L} \leftarrow \text{arc length}$$

Line integral along C with respect to x, y, and z can also be defined as

$$\boxed{\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_a^b [P(x,y,z)x'(t) + Q(x,y,z)y'(t) + R(x,y,z)z'(t)]dt}$$

Example 4. Evaluate $\int_C x^2 z ds$ if C is given by $x = \sin(2t)$, $y = 3t$, $z = \cos(2t)$, $0 \le t \le \pi/4$.

$x'(t) = 2\cos 2t$, $y'(t) = 3$, $z'(t) = -2\sin 2t$

$ds = \sqrt{[x']^2 + [y']^2 + [z']^2}\, dt = \sqrt{4\cos^2 2t + 9 + 4\sin^2 2t}\, dt = \sqrt{13}\, dt$

$\int_C x^2 z\, ds = \int_0^{\pi/4} \sin^2(2t)\cos(2t)\sqrt{13}\, dt$ $\quad$ $u = \sin(2t)$, $du = 2\cos(2t)dt$ $\quad$ $0 \to 0$, $\pi/4 \to 1$

$= \frac{\sqrt{13}}{2}\int_0^1 u^2\, du = \boxed{\frac{\sqrt{13}}{6}}$

Example 5. Evaluate $\int_C yzdy + xydz$ if C is given by $x = \sqrt{t}$, $y = t$, $z = t^2$, $0 \le t \le 1$.

$dx = \frac{1}{2}t^{-1/2}dt$, $dy = dt$, $dz = 2t\,dt$

$\int_C yzdy + xydz = \int_0^1 [t(t^2) + \sqrt{t}\, t\, 2t]\, dt$

$= \int_0^1 [t^3 + 2t^{5/2}]\, dt$

$= \left[\frac{t^4}{4} + 2\cdot\frac{2}{7}t^{7/2}\right]_0^1$

$= \boxed{\frac{1}{4} + \frac{4}{7}}$

Line integrals of vector fields.

Definition. Let $\vec{F}$ be continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then the line integral of F along C is

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)dt} = \int_C \vec{F}\cdot\vec{T}ds$$

dot product

where $\vec{T}$ is a unit tangent vector.

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C Pdx + Qdy + Rdz}$$

Example 6. Find the work done by the force field $\vec{F}(x,y,z) = xz\vec{i} + xy\vec{j} + yz\vec{k}$ on a particle that moves along the curve $\vec{r}(t) = < t^2, -t^3, t^4 >$, $0 \le t \le 1$.

$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\, dt$

$\vec{r}'(t) = < 2t, -3t^2, 4t^3 >$

$\vec{F}(\vec{r}(t)) = < t^2(t^4), t^2(-t^3), -t^3(t^4) >$

$= < t^6, -t^5, -t^7 >$

$\vec{F}(\vec{r}(t))\cdot\vec{r}'(t) = t^6(2t) - t^5(-3t^2) - t^7(4t^3)$

$= 5t^7 - 4t^{10}$

$W = \int_0^1 [5t^7 - 4t^{10}]dt = \left[\frac{5t^8}{8} - \frac{4t^{11}}{11}\right]_0^1 = \boxed{\frac{5}{8} - \frac{4}{11}}$