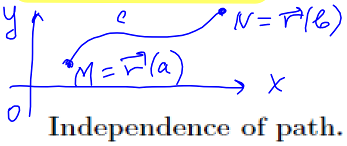
Section 14.3 **The Fundamental Theorem for line integrals.**

conservative vector fields.

Theorem. Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then



$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))} = f(N) - f(M)$$

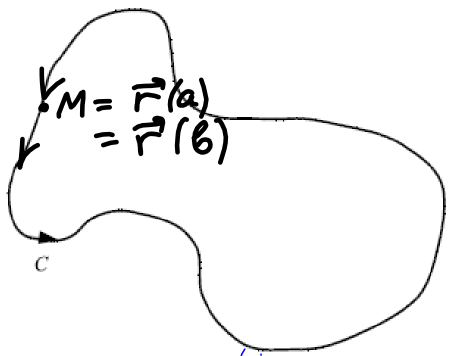
Independence of path.

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and the terminal point B. In general, $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$. But, according the the Theorem, if ∇f is continuous, then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$. In other words, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

In general, if $\vec{F}$ is a continuous vector-field with domain D, we say that the line integral is **independent of path** if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. Line integrals of conservative vector fields are independent of path.

$\int_C \vec{F} \cdot d\vec{r}$ make sure that $\vec{F}$ is conservative.

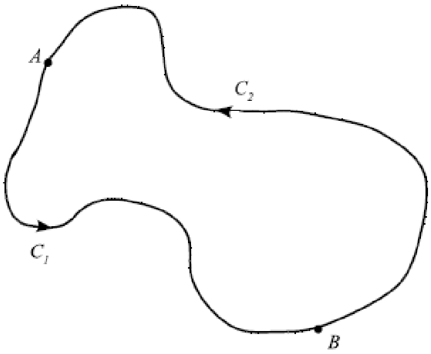
A curve is called closed if its terminal point coincides with its initial point, that is $\vec{r}(a) = \vec{r}(b)$.



$\vec{F}$ is conservative if $\oint_C \vec{F} \cdot d\vec{r} = 0$

$\oint_C$ – line integral along a closed curve.

If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A.



Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

Also we can show that if $\oint_C \vec{F} \cdot d\vec{r} = 0$ whenever C is a closed path in D, then $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D.

Theorem. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path in D.

Now we assume that D is **open** (for every point P in D there is a disk with center P that lies entirely in D) and **connected** (any two points in D can be joined by a path that lies in D).

Theorem. Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \vec{F}$.

Question: How to determine whether or not a vector field F is conservative?

Theorem. If $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ is a conservative vector fields, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem is true only for a special type of the region.

Definition. A curve is **simple** if it does not cross itself anywhere between its endpoints.

Definition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D (simply-connected region contains no hole and cannot consist of two separate pieces).

Theorem. Let $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Then $\vec{F}$ is conservative.

Example 1. Determine whether or not the vector field

$$\vec{F}(x,y) = (\underbrace{y\cos x - \cos y}_{P(x,y)})\vec{\imath} + (\underbrace{\sin x + x\sin y}_{Q(x,y)})\vec{\jmath}$$

is conservative.

$P(x,y) = y\cos x - \cos y$ $\qquad$ $Q(x,y) = \sin x + x\sin y$

$\frac{\partial P}{\partial y} = \cos x + \sin y = \frac{\partial Q}{\partial x} = \cos x + \sin y$

$\boxed{\text{YES}}$

Example 2

1. If $\vec{F} = < 2xy^3, 3x^2y^2 >$, find a function f such that $\nabla f = \vec{F}$.

F is conservative.

$f(x,y)$ such that $\nabla f = \vec{F}$

$< f_x, f_y > = < 2xy^3, 3x^2y^2 >$

$\int f_x\,dx = f(x,y) + h(y)$

$\int f_y\,dy = f(x,y) + g(x)$

$\begin{cases} f_x = 2xy^3 \\ f_y = 3x^2y^2 \end{cases}$

$\int f_x \,dx = \int 2xy^3\,dx$

$f(x,y) = x^2y^3 + h(y)$, h is an unknown function.

Plug $f(x,y)$ into the 2nd equation.

Take $\frac{\partial}{\partial y} f(x,y)$: $f_y = 3x^2y^2 + h'(y)$

$\cancel{3x^2y^2} + h'(y) = \cancel{3x^2y^2}$

$h'(y) = 0$, $h(y) = C$ (constant)

we can plug $C = 0$.

$\boxed{f(x,y) = x^2y^3}$

2. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ along the curve C given by $\vec{r}(t) =< \sin t, t^2+1 >, 0 \le t \le \pi/2$.

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\frac{\pi}{2})) - f(\vec{r}(0))$

where $f(x,y) = x^2y^3$

$\vec{r}(t) = < \sin t, t^2+1 >$

$\vec{r}(\frac{\pi}{2}) = < \sin\frac{\pi}{2}, \frac{\pi^2}{4}+1 >$

$= < 1, \frac{\pi^2}{4}+1 >$

$\vec{r}(0) = < \sin 0, 1 >$

$= < 0, 1 >$

$f(1, \frac{\pi^2}{4}+1) = (\frac{\pi^2}{4}+1)^3$

$f(0,1) = 0$

$\int_C \vec{F}\cdot d\vec{r} = \boxed{\left(\frac{\pi^2}{4}+1\right)^3}$

Example 3

1. If $\vec{F} =< 2xz + \sin y, x\cos y, x^2 >$, find a function f such that $\nabla f = \vec{F}$.

$\nabla f = F$

$< f_x, f_y, f_z > = < 2xz + \sin y, x\cos y, x^2 > \rightarrow \begin{cases} f_x = 2xz + \sin y \\ f_y = x\cos y \\ f_z = x^2 \end{cases}$

$\int f_z dz = \int x^2 dz$

$f(x,y,z) = x^2 z + g(x,y)$ g is unknown

plug f into the 2nd equation.

$f_y = g_y$

$\int g_y dy = \int x\cos y\, dy$

$g(x,y) = x\sin y + h(x)$ h is unknown.

$f(x,y,z) = x^2 z + x\sin y + h(x)$

plug f into the 1st equation

$f_x = 2xz + \sin y + h'(x)$

$2xz + \sin y + h'(x) = 2xz + \sin y$

$h'(x) = 0$

$h(x) = C$ (constant)

we can plug $C = 0$.

$\boxed{f(x,y,z) = x^2 z + x\sin y}$

2. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ along the curve C given by $\vec{r}(t) =< \cos t, \sin t, t >$, $0 \le t \le 2\pi$.

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2\pi)) - f(\vec{r}(0))$

$f(x,y,z) = x^2 z + x\sin y$

$\vec{r}(t) = < \cos t, \sin t, t >$

$\vec{r}(2\pi) = < 1, 0, 2\pi >$ | $f(1,0,2\pi) = 2\pi$

$\vec{r}(0) = < 1, 0, 0 >$ | $f(1,0,0) = 0$

$\int_C \vec{F} \cdot d\vec{r} = \boxed{2\pi}$