Section 14.5 **Curl and divergence.**

Curl.
If $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the curl of $\vec{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{\operatorname{curl} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}}$$

Let $\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ be the vector differential operator. $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, f is a function,

$$\nabla f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$$
$$= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \text{grad } f.$$

Then

$$\boxed{\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \overbrace{\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)}^{-\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)}\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \operatorname{curl} \vec{F}.}$$

Example 1. Find curl $\vec{F}$ if $\vec{F}(x,y,z) = \underbrace{xe^y}_{P}\vec{\imath} - \underbrace{ze^{-y}}_{Q}\vec{\jmath} + \underbrace{y\ln(z)}_{R}\vec{k}$.
$= \langle xe^y, -ze^{-y}, y\ln z \rangle.$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y & -ze^{-y} & y\ln z \end{vmatrix} = \vec{\imath}\left(\underbrace{\frac{\partial}{\partial y}(y\ln z)}_{\ln z} - \underbrace{\frac{\partial}{\partial z}(-ze^{-y})}_{-e^{-y}}\right)$$
$$-\vec{\jmath}\left(\underbrace{\frac{\partial}{\partial x}(y\ln z)}_{0} - \underbrace{\frac{\partial}{\partial z}(xe^y)}_{0}\right)$$
$$+\vec{k}\left(\underbrace{\frac{\partial}{\partial x}(-ze^{-y})}_{0} - \underbrace{\frac{\partial}{\partial y}(xe^y)}_{xe^y}\right)$$
$$= \boxed{\langle \ln z + e^{-y}, 0, -xe^y \rangle}$$

Theorem 1. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\boxed{\begin{gathered}\operatorname{curl}(\text{grad } f) = \vec{0} \\ \operatorname{curl}(\nabla f) = \vec{0}\end{gathered}} \text{ for all } f.$$

Theorem 2. If $\vec{F}$ is a vector field defined on all on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

Example 2. Determine whether or not the vector field $\vec{F} = zx\vec{\imath} + xy\vec{\jmath} + yz\vec{k}$ is conservative. If it is conservative, find a function f such that $\vec{F} = \nabla f$. $= \langle zx, xy, yz \rangle$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zx & xy & yz \end{vmatrix} = \vec{\imath}\left(\underbrace{\frac{\partial}{\partial y}(yz)}_{z} - \underbrace{\frac{\partial}{\partial z}(xy)}_{0}\right)$$
$$-\vec{\jmath}\left(\underbrace{\frac{\partial}{\partial x}(yz)}_{0} - \underbrace{\frac{\partial}{\partial z}(zx)}_{x}\right)$$
$$+\vec{k}\left(\underbrace{\frac{\partial}{\partial x}(xy)}_{y} - \underbrace{\frac{\partial}{\partial y}(zx)}_{0}\right)$$
$$= \langle z, +x, y \rangle \neq \langle 0, 0, 0 \rangle$$

$$\boxed{\text{NOT CONSERVATIVE}}$$

The curl vector is associated with rotation. If for a vector field $\vec{F}$ curl $\vec{F} = \vec{0}$, then the field $\vec{F}$ is irrotational.

Divergence.

Definition. If $\vec{F} =< P, Q, R >$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z exist, then the **divergence** of $\vec{F}$ is the function of tree variables defined by

$$\boxed{\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

P Q R

Example 3. Find the divergence of the vector field $\vec{F}(x, y, z) = xe^y\vec{\imath} - ze^{-y}\vec{\jmath} + y\ln(z)\vec{k}$.

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(-ze^{-y}) + \frac{\partial}{\partial z}(y \ln z)$$

$$= \boxed{e^y + ze^{-y} + \frac{y}{z}}$$

Theorem 3. If $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + Q\vec{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order derivatives, then

$$\text{div curl } \vec{F} = 0 \quad \text{allways!}$$

Example 4. Show that there is no vector field $\vec{G}$ such that curl$\vec{G} = 2x\vec{\imath} + 3yz\vec{\jmath} + xz^2\vec{k}$

$$\text{curl } \vec{F} = \left< \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right>$$

$$\text{div}(\text{curl } \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)$$

$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}} - \cancel{\frac{\partial^2 R}{\partial x \partial y}} + \cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 P}{\partial z \partial y}} + \cancel{\frac{\partial^2 Q}{\partial x \partial z}} = 0$$

Proof of Thm. 3

$$\text{div}(\text{curl } \vec{G}) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3yz) + \frac{\partial}{\partial z}(xz^2)$$

$$= 2 + 3z + 2xz \neq 0$$

Divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. If div $\vec{F} = 0$, then $\vec{F}$ is said to be incompressible.

Laplace operator:

$$\boxed{\operatorname{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f}$$

If f is a scalar field and $\vec{F}$, $\vec{G}$ are vector fields, then $f\vec{F}$, $\vec{F} \cdot \vec{G}$, and $\vec{F} \times \vec{G}$ are vector fields defined by

$$(f\vec{F})(x, y, z) = f(x, y, z)\vec{F}(x, y, z)$$
$$(\vec{F} \cdot \vec{G})(x, y, z) = \vec{F}(x, y, z) \cdot \vec{G}(x, y, z)$$
$$(\vec{F} \times \vec{G})(x, y, z) = \vec{F}(x, y, z) \times \vec{G}(x, y, z)$$

and

1. $\operatorname{div}(\vec{F} + \vec{G}) = \operatorname{div} \vec{F} + \operatorname{div} \vec{G}$
2. $\operatorname{curl}(\vec{F} + \vec{G}) = \operatorname{curl} \vec{F} + \operatorname{curl} \vec{G}$
3. $\operatorname{div}(f\vec{F}) = f\operatorname{div} \vec{F} + \vec{F} \cdot \nabla f$
4. $\operatorname{curl}(f\vec{F}) = f\operatorname{curl} \vec{F} + (\nabla f) \times \vec{F}$
5. $\operatorname{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \operatorname{curl} \vec{F} - \vec{F} \cdot \operatorname{curl} \vec{G}$
6. $\operatorname{div}(\nabla f \times \nabla g) = 0$
7. $\operatorname{curl}\operatorname{curl}(\vec{F}) = \operatorname{grad}\operatorname{div} \vec{F} - \nabla^2 \vec{F}$
8. $\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F} + \vec{F} \times \operatorname{curl} \vec{G} + \vec{G} \times \operatorname{curl} \vec{F}$

Vector forms of Green's Theorem.

Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field. We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P dx + Q dy$$

and

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

Therefore $$(\text{curl } \vec{F}) \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can rewrite the equation in Green's Theorem in the vector form

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k} \, dA}$$

If C is given by the vector equation $\vec{r}(t) =< x(t), y(t) >$, $a \le t \le b$, then the unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$$

Then the outward unit normal vector to C is given by

$$\vec{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}\vec{j}$$

Then

$$\oint_C \vec{F}\vec{n} \, ds = \int_a^b (\vec{F} \cdot \vec{n})(t) |\vec{r}'(t)| dt$$

$$= \int_C P dy - Q dx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

So,

$$\boxed{\oint_C \vec{F} \cdot \vec{n} ds = \iint_D \text{div } \vec{F}(x,y) dA}$$