Chapter 14. **Vector calculus.**
Section 14.6 **Parametric surfaces and their areas.**

We suppose that

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}, \quad (u,v) \in \mathcal{D} \text{ in } \mathbb{R}^2$$

is a vector-valued function defined on an region D in the uv-plane and the partial derivatives of x, y, and z with respect to u and v are all continuous. The set of all points $(x,y,z) \in \mathbb{R}^3$, such that

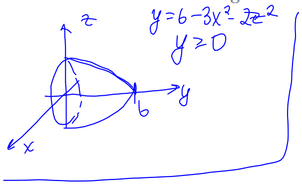
$$x = x(u,v), \quad y = y(u,v) \quad z = z(u,v)$$

and $(u,v) \in D$, is called a parametric surface S with parametric equations

$$x = x(u,v), \quad y = y(u,v) \quad z = z(u,v).$$

$\mathcal{D}$ is called the parameter domain for S.

Example 1. Find a parametric representation for the part of the elliptic paraboloid $y = 6 - 3x^2 - 2z^2$ that lies to the right of the xz-plane.

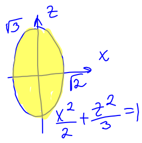


1. $\begin{cases} x = x \\ y = 6-3x^2-2z^2 \\ z = z \end{cases}$ parametrization.

Parameter domain: $6-3x^2-2z^2 \geq 0$

$3x^2+2z^2 \leq 6$

$\frac{x^2}{2} + \frac{z^2}{3} \leq 1$



2. "cylindrical coordinates"

$\begin{cases} x = \sqrt{2}\, r\cos\theta \\ y = 6 - 3(\sqrt{2}\, r\cos\theta)^2 - 2(\sqrt{3}\, r\sin\theta)^2 = 6-6r^2 \\ z = \sqrt{3}\, r\sin\theta \end{cases}$

Parameter domain: $0 \leq \theta \leq 2\pi$

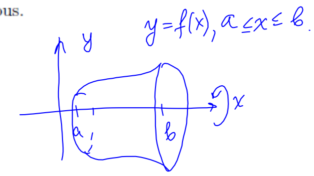
$6-6r^2 \geq 0$

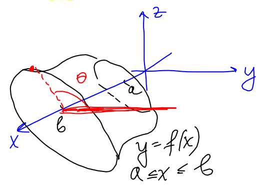
$r^2 \leq 1, \quad 0 \leq r \leq 1$

In general, a surface given as the graph of the function $z = z(x,y)$, can always be regarded as a parametric surface with parametric equations

$$\boxed{x = x,} \; \boxed{y = y} \; \boxed{z = z(x,y).}$$

Surfaces of revolution also can be represented parametrically. Let us consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$ and f' is continuous.





Let θ be the angle of rotation. If (x,y,z) is a point on S, then

$$\boxed{x = x} \; \boxed{y = f(x)\cos\theta} \; \boxed{z = f(x)\sin\theta}$$

The parameter domain is given by $a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.

Example 2. Find equation for the surface generated by rotating the curve $x = 4y^2 - y^4$, $-2 \leq y \leq 2$, about the y-axis.

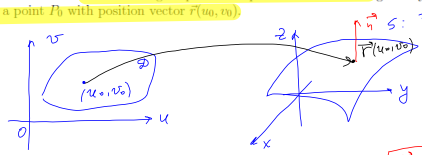
θ is the angle of rotation.

$\begin{cases} x = (4y^2-y^4)\cos\theta \\ y = y \\ z = (4y^2-y^4)\sin\theta \end{cases}$

Parameter domain: $-2 \leq y \leq 2$
$0 \leq \theta \leq 2\pi$

Tangent planes.

Problem. Find the tangent plane to a parametric surface S given by a vector function $\vec{r}(u,v)$ at a point P_0 with position vector $\vec{r}(u_0,v_0)$.



$\vec{n}$ is the normal vector to the tangent plane.

$\boxed{\vec{n} = \vec{r}_u(u_0,v_0)\times\vec{r}_v(u_0,v_0)}$

If S is given by $z=z(x,y)$. Then the normal vector to the tangent plane to the surface S at $(x_0,y_0,z(x_0,y_0))$

$\boxed{\vec{n} = \langle z_x, z_y, -1\rangle}$

The tangent vector $\vec{r}_v$ to C_1 at P_0 is

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\vec{\imath} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{\jmath} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$$

Similarly, the tangent vector $\vec{r}_u$ to C_2 at P_0 is

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\vec{\imath} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{\jmath} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$$

Then the **normal vector** to the tangent plane to a parametric surface S at P_0 is the vector $\vec{r}_u\times\vec{r}_v$. If $\vec{r}_u\times\vec{r}_v\neq\vec{0}$, then S is called **smooth**.

Example 3. Find the tangent plane to the surface with parametric equations $\vec{r}(u,v) = (u+v)\vec{\imath} + u\cos v\vec{\jmath} + v\sin u\vec{k}$ at the point (1,1,0).

$\vec{r}(u,v) = \langle u+v, u\cos v, v\sin u\rangle$ @ $(1,1,0)$

Find (u_0,v_0) such that

$\vec{r}(u_0,v_0) = \langle 1,1,0\rangle$

$1\cos 0 = 1$ $\quad 0\sin 1 = 0$

$\vec{r}(u_0,v_0) = \langle \underset{1}{u_0} + \underset{0}{v_0}, u_0\cos v_0, v_0\sin u_0\rangle = \langle 1,1,0\rangle$

$\langle 1,1,0\rangle$

$u_0 = 1, \quad v_0 = 0$

Normal vector $\quad n = \vec{r}_u(u_0,v_0)\times\vec{r}_v(u_0,v_0)$

$\vec{r}_u = \langle 1, \cos v, v\cos u\rangle \quad \vec{r}_u(1,0) = \langle 1,1,0\rangle$

$\vec{r}_v = \langle 1, -u\sin v, \sin u\rangle \quad \vec{r}_v(1,0) = \langle 1, 0, \sin 1\rangle$

$\vec{n} = \langle 1,1,0\rangle\times\langle 1,0,\sin 1\rangle$

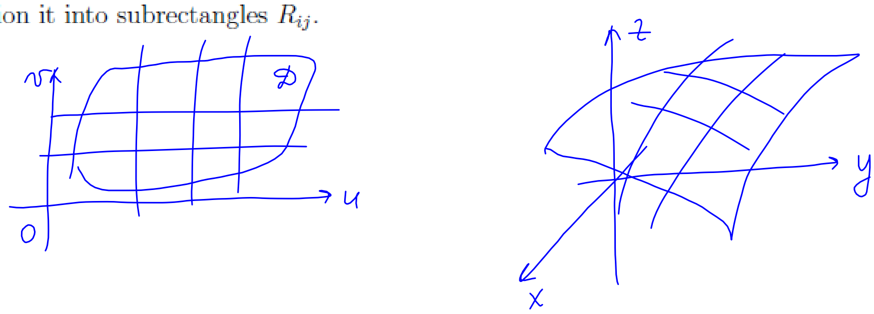
$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & \sin 1 \end{vmatrix} = \langle \sin 1, -\sin 1, -1\rangle$$

Tangent plane

$\boxed{\sin 1(x-1) - \sin 1(y-1) - 1(z-0) = 0}$

Surface area. Let S be a parametric surface given by a vector function $\vec{r}(u,v)$, $(u,v)\in D$. For simplicity, we start by considering a surface whose parameter domain D is a rectangle, and we partition it into subrectangles R_{ij}.

$S: \vec{r}(u,v)$
$(u,v)\in\mathcal{D}$



Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. The part S_{ij} of the surface S that corresponds to R_{ij} has the point P_{ij} with position vector $\vec{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\vec{r}_{u_i} = \vec{r}_u(u_i^*, v_j^*), \qquad \vec{r}_{v_j} = \vec{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij}. We approximate S_{ij} by the parallelogram determined by the vectors $\Delta u_i\vec{r}_{u_i}$ and $\Delta v_j\vec{r}_{v_j}$ (this parallelogram lies in the tangent plane to S at P_{ij}). The area of this parallelogram is

$$|(\Delta u_i\vec{r}_{u_i})\times(\Delta v_j\vec{r}_{v_j})| = |\vec{r}_{u_i}\times\vec{r}_{v_j}|\Delta u_i\Delta v_j$$

so an approximation to the area of S is

$$\sum_{i=1}^{m}\sum_{j=1}^{n}|\vec{r}_{u_i}\times\vec{r}_{v_j}|\Delta u_i\Delta v_j \to \iint_D |\vec{r}_u\times\vec{r}_v|\,dA \text{ as } \|P\|\to 0.$$

Definition. If a smooth parametric surface S is given by the equation $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, $(u,v) \in D$ and S is covered just once at (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$\boxed{A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

magnitude of the normal vector.

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k} \qquad \vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

If a surface S is given by $z = z(x,y)$, $(x,y) \in D$, the parametric equations for S are

$$x = x, \quad y = u \quad z = z(x,y)$$

Then $\vec{r}_x = < 1, 0, z_x(x,y) >$, $\vec{r}_y = < 0, 1, z_y(x,y) >$, and

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & z_x(x,y) \\ 0 & 1 & z_y(x,y) \end{vmatrix} = -z_x\vec{i} - z_y\vec{j} + \vec{k}$$

Then

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + [z_x(x,y)]^2 + [z_y(x,y)]^2}$$

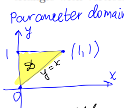
and

$$\boxed{A(S) = \iint_D \sqrt{1 + [z_x(x,y)]^2 + [z_y(x,y)]^2} \, dA}$$

3

Example 4. Find the surface area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1), and (0,1).

Parameter domain



$0 \le x \le y$

$0 \le y \le 1$

$z = x + y^2$, $z_x = 1$, $z_y = 2y$

$$A = \iint_D \sqrt{1 + [z_x]^2 + [z_y]^2} \, dA$$

$$= \iint_D \sqrt{1 + 1 + 4y^2} \, dA$$

$$= \int_0^1 \int_0^y \sqrt{2 + 4y^2} \, dx \, dy$$

$$= \frac{1}{8} \int_0^1 8y \sqrt{2 + 4y^2} \, dy$$

$u = 2 + 4y^2$

$du = 8y \, dy$

$0 \to 2$

$1 \to 6$

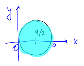
$$= \frac{1}{8} \int_2^6 \sqrt{u} \, du$$

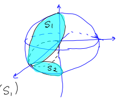
$$= \frac{1}{8} \cdot \frac{2}{3} \, u^{3/2} \Big]_2^6$$

$$= \boxed{\frac{1}{12}\left(6\sqrt{6} - 2\sqrt{2}\right)}$$

Example 5. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$

$(x - \frac{1}{2}a)^2 + y^2 = \frac{a^2}{4}$





$A = A(S_1) + A(S_2) = 2A(S_1)$

Surface S_1

Step 1. Paramertize S_1:

spherical coordinates:

$$\begin{cases} x = \rho \cos\theta \sin\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\varphi \end{cases}$$

Plug x, y, z into the equation of the sphere

$\underbrace{x^2 + y^2 + z^2}_{\rho^2} = a^2$

$\rho^2 = a^2$, $\rho = a$

Parametric equations of the sphere:

$$\begin{cases} x = a \cos\theta \sin\varphi \\ y = a \sin\theta \sin\varphi \\ z = a \cos\varphi \end{cases}$$

Step 2. Parameter domain for S_1.

$$\left(\begin{matrix} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le \varphi \le \pi/2 \end{matrix}\right) = D$$

Step 3. Find $|\vec{r}_\theta \times \vec{r}_\varphi|$

$\vec{r}(\theta, \varphi) = < a\cos\theta\sin\varphi, a\sin\theta\sin\varphi, a\cos\varphi >$

$\vec{r}_\theta = < -a\sin\theta\sin\varphi, a\cos\theta\sin\varphi, 0 >$

$\vec{r}_\varphi = < a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, -a\sin\varphi >$

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi & 0 \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi & -a\sin\varphi \end{vmatrix}$$

$$= < -a^2\cos\theta\sin^2\varphi, -a^2\sin\theta\sin^2\varphi, \underbrace{-a^2\sin^2\theta\sin\varphi\cos\varphi - a^2\cos^2\theta\sin\varphi\cos\varphi}_{-a^2\sin\varphi\cos\varphi(\sin^2\theta + \cos^2\theta)} >$$

$$= -a^2 < \cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, \sin\varphi\cos\varphi >$$

$$|\vec{r}_\theta \times \vec{r}_\varphi| = a^2\sqrt{\underbrace{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi}_{\sin^4\varphi(\cos^2\theta + \sin^2\theta)} + \sin^2\varphi\cos^2\varphi}$$

$$= a^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$

$$= a^2\sqrt{\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$$

$$= a^2\sin\varphi$$

Step 4. Find the area.

$$A = 2\iint_D a^2\sin\varphi \, dA$$

$$= 2\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2} a^2\sin\varphi \, d\varphi \, d\theta$$

$$= 2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) a^2(-\cos\varphi)\Big|_0^{\pi/2}$$

$$= \boxed{2\pi a^2}$$