Chapter 14. **Vector calculus.**
Section 14.7 **Surface integrals.**

Suppose f is a function of three variables whose domain include a surface S. We divide S into patches S_{ij} with area ΔS_{ij}. We evaluate f at a point P^*_{ij} in each patch, multiply by the area ΔS_{ij}, and form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P^*_{ij})\Delta S_{ij}$$

We define the **surface integral of** f **over the surface** S as

$$\iint_S f(x,y,z)dS = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P^*_{ij})\Delta S_{ij}$$

If the surface S is given by an equation $z = z(x,y)$, $(x,y)\in D$, then

$$\boxed{\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y))\sqrt{\left[\frac{\partial z}{\partial x}\right]^2+\left[\frac{\partial z}{\partial y}\right]^2+1}\,dA}$$

If the surface S is given by vector function $\vec r(u,v) = x(u,v)\vec\imath + y(u,v)\vec\jmath + z(u,v)\vec k$, $(u,v)\in D$, then

$$\boxed{\iint_S f(x,y,z)dS = \iint_D f(\vec r(u,v))|\vec r_u\times\vec r_v|\,dA}$$

where

$$\vec r_u = \frac{\partial x}{\partial u}\vec\imath + \frac{\partial y}{\partial u}\vec\jmath + \frac{\partial z}{\partial u}\vec k \qquad \vec r_v = \frac{\partial x}{\partial v}\vec\imath + \frac{\partial y}{\partial v}\vec\jmath + \frac{\partial z}{\partial v}\vec k$$

Example 1.

1. Evaluate $\iint_S y\,dS$, where S is the part of the plane $3x+2y+z=6$ that lies in the first octant.

$z = 6-3x-2y$

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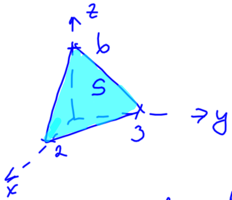
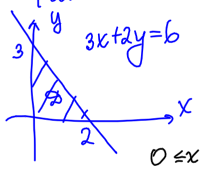


Parameter domain.



$0 \le x \le \frac{6-2y}{3}$
$0 \le y \le 3$

$$\iint_S y\,dS = \iint_D y\sqrt{1+[z_x]^2+[z_y]^2}\,dA$$

$z_x = -3,\quad z_y=-2$

$$= \int_0^3\int_0^{\frac{6-2y}{3}} y\sqrt{1+9+4}\,dx\,dy$$

$$= \frac{\sqrt{14}}{3}\int_0^3 \overbrace{y(6-2y)}^{6y-2y^2}dy$$

$$= \frac{\sqrt{14}}{3}\left(\frac{6y^2}{2} - 2\frac{y^3}{3}\right)\Big|_0^3$$

$$= \frac{\sqrt{14}}{3}(27-18)$$

$$= \boxed{3\sqrt{14}}$$

$= \langle u\cos v, u\sin v, v\rangle$

2. Evaluate $\iint_S \sqrt{1+x^2+y^2}\,dS$, if S is given by vector equation $\vec{r}(u,v) = u\cos v\vec{\imath} + u\sin v\vec{\jmath} + v\vec{k}$,
$D = \{0 \le u \le 1,\ 0 \le v \le \pi\}$ – parameter domain

$$\iint_S \sqrt{1+x^2+y^2}\,dS = \iint_D \underbrace{\sqrt{1+u^2\cos^2 v + u^2\sin^2 v}}_{1+u^2}\ |\vec{r}_u \times \vec{r}_v|\,dA$$

$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \vec{\imath}(\sin v) - \vec{\jmath}(\cos v) + \vec{k}(u\cos^2 v + u\sin^2 v)$$

$$= \langle \sin v, -\cos v, u\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2}$$
$$= \sqrt{1+u^2}$$

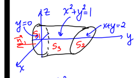
$$\iint_S \sqrt{1+x^2+y^2}\,dS = \int_0^1 \int_0^\pi \sqrt{1+u^2}\sqrt{1+u^2}\,dv\,du$$

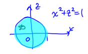
$$= \pi \int_0^1 (1+u^2)\,du$$

$$= \pi\left(u + \frac{u^3}{3}\right)\Big|_0^1$$

$$= \boxed{\frac{4\pi}{3}}$$

3. Evaluate $\iint_S xy\,dS$, if S is a boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=0$ and $x+y=2$.





$$\iint_S xy\,dS = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

S_1: $y=0$

$$\iint_S xy\,dS = \iint_D x\underbrace{(0)}_{y \text{ on } S_1}|\vec{n}|\,dA = 0$$

$\vec{n}_1 = \langle 0,-1,0\rangle$, $|\vec{n}_1| = 1$

S_2: $x+y=2$

Parameter domain $[x^2+z^2\le 1]$

Param. equations:

$$\begin{cases} x = x \\ y = 2-x \\ z = z \end{cases}$$

Vector equation of S_2: $\vec{r}(x,z) = \langle x, 2-x, z\rangle$

$\vec{r}_x = \langle 1,-1,0\rangle$

$\vec{r}_z = \langle 0,0,1\rangle$

$$\vec{r}_x\times\vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\vec{i}-\vec{j} = \langle -1,-1,0\rangle$$

$|\vec{r}_x\times\vec{r}_z| = \sqrt{2}$

$$\iint_{S_2} xy\,dS = \iint_D x(2-x)|\vec{r}_x\times\vec{r}_z|\,dA$$

$$= \sqrt{2}\iint_D [2x-x^2]\,dA$$

Polar coordinates:

$x = r\cos\theta$

$z = r\sin\theta$

$dA = r\,dr\,d\theta$

$0\le\theta\le 2\pi$

$0\le r\le 1$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 [2r\cos\theta - r^2\cos^2\theta]\,r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 [2r^2\cos\theta - r^3\cos^2\theta]\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{2r^3}{3}\cos\theta - \frac{r^4}{4}\cos^2\theta\right]_{r=0}^{r=1}d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\cancel{\frac{2}{3}\cos\theta}^{\,0} - \frac{1}{4}\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\right]d\theta$$

$$= -\frac{\sqrt{2}}{8}\int_0^{2\pi}(1+\cancel{\cos 2\theta}^{\,0})\,d\theta$$

$$= -\frac{2\pi\sqrt{2}}{8}$$

$$= \boxed{-\frac{\sqrt{2}\pi}{4}}$$

S_3: lateral surface of the cylinder $x^2+z^2=1$, $0\le y\le 2-x$

$$\iint_{S_3} xy\,dS = \iint_{D_1} xy\,|\vec{n}|\,dA$$

Param. equations of S_3.

cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = y \\ z = r\sin\theta \end{cases}$$

Plug x and z into $x^2+z^2=1$

$r^2=1$ or $r=1$

Parametric equations of the cylinder:

$$\begin{cases} x = \cos\theta \\ y = y \\ z = \sin\theta \end{cases}$$

Parameter domain: $\begin{pmatrix} 0\le\theta\le 2\pi \\ 0\le y\le \underset{2-\cos\theta}{2-x} \end{pmatrix} = D_1$

Vector equation of S_3:

$\vec{r}(\theta,y) = \langle\cos\theta, y, \sin\theta\rangle$

$\vec{r}_\theta = \langle -\sin\theta, 0, \cos\theta\rangle$

$\vec{r}_y = \langle 0,1,0\rangle$

$$\vec{r}_\theta\times\vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix} = -\cos\theta\,\vec{i} - \sin\theta\,\vec{k} = \langle -\cos\theta, 0, -\sin\theta\rangle$$

$|\vec{r}_\theta\times\vec{r}_y| = 1$

$$\iint_{S_3} = \int_0^{2\pi}\int_0^{2-\cos\theta} \overbrace{\cos\theta}^{x}\,y\,dy\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\cos\theta\,\overbrace{(2-\cos\theta)^2}^{4-4\cos\theta+\cos^2\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(\cancel{4\cos\theta}^{\,0} - 4\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= -2\int_0^{2\pi}\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\,d\theta + \cancel{\frac{1}{2}\int_0^{2\pi}\cos^3\theta\,d\theta}^{\,0}$$

$\cos\theta\cos^2\theta$

$\cos\theta\,(1-\sin^2\theta)$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$0 \to \sin 0 = 0$

$2\pi \to \sin 2\pi = 0$

$$= -\int_0^{2\pi}(1+\cancel{\cos 2\theta}^{\,0})\,d\theta$$

$$= \boxed{-2\pi}$$

$$\iint_S = \boxed{-\frac{\sqrt{2}\pi}{4} - 2\pi}$$

If a thin sheet has the shape of a surface S and the density at the point (x, y, z) is $\rho(x, y, z)$, then the total mass of the sheet is

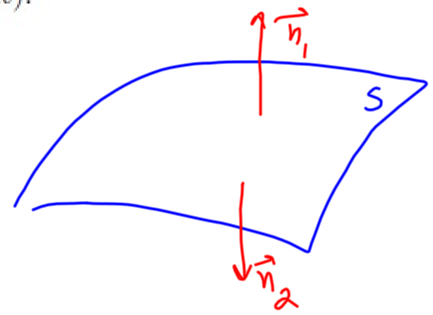
$$\boxed{m = \iint_S \rho(x, y, z) dS}$$

and the center of mass is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{\bar{x} = \frac{1}{m} \iint_S x\rho(x, y, z) dS} \quad \boxed{\bar{y} = \frac{1}{m} \iint_S y\rho(x, y, z) dS} \quad \boxed{\bar{z} = \frac{1}{m} \iint_S z\rho(x, y, z) dS}$$

Oriented surfaces.

Let us consider a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).



$\vec{n}_1 = -\vec{n}_2$

There are two unit normal vectors $\vec{n_1}$ and $\vec{n_2} = -\vec{n_1}$ at (x, y, z). If it is possible to chose a unit normal vector $\vec{n}$ at every such point (x, y, z) so that $\vec{n}$ varies continuously over S, then S is called an oriented surface and the given choice of $\vec{n}$ provides S with an orientation. There are two possible orientations for any orientable surface.

For a surface $z = z(x, y)$ the orientation is given by the unit normal vector

$$\vec{n} = \frac{-\frac{\partial z}{\partial x}\vec{\imath} - \frac{\partial z}{\partial y}\vec{\jmath} + \vec{k}}{\sqrt{\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 + 1}}$$

$z = z(x, y)$

$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$

if $\vec{n}$ is directed upward, then

$\boxed{\vec{n} = -\langle z_x, z_y, -1 \rangle}$

Since the $\vec{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\vec{r}(u, v)$, then its orientation is given by a unit normal vector

$\vec{n} = \pm \vec{r}_u \times \vec{r}_v$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

For a closed surface, the positive orientation is the one for which the normal vectors point *outward* from S, the inward-pointing normals give the negative orientation.

Surface integrals of vector fields.

Definition. If $\vec{F}$ is a continuous vector-field defined on an oriented surface S with normal vector $\vec{n}$, then the surface integral of F over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \vec{n}\, dA$$

4

This integral is also called the flux of $\vec{F}$ across S.

If $\vec{F} =< P(x,y,z), Q(x,y,z), R(x,y,z) >$ and the surface S is given by an equation $z = g(x,y)$, $(x,y) \in D$, then

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{\imath} - \frac{\partial g}{\partial y}\vec{\jmath} + \vec{k}}{\sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1}}$$

and

$$\iint_S \vec{F} \cdot dS = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D (P\vec{\imath} + Q\vec{\jmath} + R\vec{k}) \cdot \frac{-\frac{\partial g}{\partial x}\vec{\imath} - \frac{\partial g}{\partial y}\vec{\jmath} + \vec{k}}{\sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1}} \sqrt{\left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2 + 1}\, dA$$

or

$$\iint_S \vec{F} \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

If the surface S is given by vector function $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$, $(u,v) \in D$, then

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and

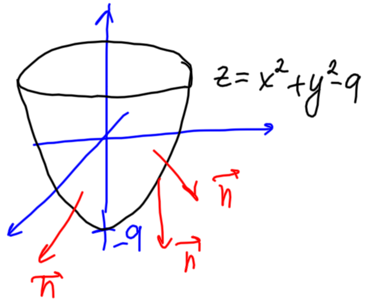
$$\iint_S \vec{F} \cdot dS = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v|\, dA$$

or

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$= \langle x^2y, -3xy^2, 4y^3 \rangle$

Example 2. Find the flux of the vector field $\vec{F} = x^2y\vec{\imath} - 3xy^2\vec{\jmath} + 4y^3\vec{k}$ across the surface S, if S is the part of the elliptic paraboloid $z = x^2 + y^2 - 9$ that lies below the rectangle $\underbrace{0 \le x \le 2,\ 0 \le y \le 1}_{= \mathcal{D}}$ and has downward orientation.



$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_{\mathcal{D}} \vec{F} \cdot \vec{n}\, dA$$

$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$$
$$= + \langle 2x, 2y, -1 \rangle$$
$$= \langle 2x, 2y, -1 \rangle$$

$$\vec{F} \cdot \vec{n} = \langle x^2y, -3xy^2, 4y^3 \rangle \cdot \langle 2x, 2y, -1 \rangle$$
$$= 2x^3y - 6xy^3 - 4y^3$$

$$\text{flux} = \int_0^2 \int_0^1 (2x^3y - 6xy^3 - 4y^3)\, dy\, dx$$

$$= \int_0^2 \left[x^3y^2 - \frac{6xy^4}{4} - y^4 \right]_{y=0}^{y=1} dx$$

$$= \int_0^2 \left[x^3 - \frac{3}{2}x - 1 \right] dx$$

$$= \left[\frac{x^4}{4} - \frac{3}{2}\frac{x^2}{2} - x \right]_0^2$$

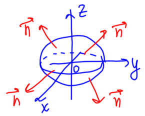
$$= 4 - 3 - 2$$
$$= \boxed{-1}$$

Example 3. A fluid has density 1500 and velocity field

$$\vec{v} = -y\vec{\imath} + x\vec{\jmath} + 2z\vec{k} = \langle -y, x, 2z \rangle$$

positive orientation

Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.



$$\text{rate of flow} = 1500 \iint_S \vec{v} \cdot d\vec{S}$$

Parametrize the sphere.

spherical coordinates:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

Plug x, y, z into $x^2 + y^2 + z^2 = 25$

$$\rho^2 = 25$$
$$\rho = 5$$

Vector of the sphere:

$$\vec{r}(\theta, \varphi) = \langle 5\cos\theta\sin\varphi,\ 5\sin\theta\sin\varphi,\ 5\cos\varphi \rangle$$

Parameter domain: $(0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi) = D$

$$\vec{n} = \pm\, \vec{r}_\theta \times \vec{r}_\varphi$$

$$\vec{r}_\theta = \langle -5\sin\theta\sin\varphi,\ 5\cos\theta\sin\varphi,\ 0 \rangle$$

$$\vec{r}_\varphi = \langle 5\cos\theta\cos\varphi,\ 5\sin\theta\cos\varphi,\ -5\sin\varphi \rangle$$

$$\vec{n} = \vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -5\sin\theta\sin\varphi & 5\cos\theta\sin\varphi & 0 \\ 5\cos\theta\cos\varphi & 5\sin\theta\cos\varphi & -5\sin\varphi \end{vmatrix}$$

$$= \vec{\imath}(-25\cos\theta\sin^2\varphi) - \vec{\jmath}(25\sin\theta\sin^2\varphi)$$
$$+ \vec{k}\big(\underbrace{-25\sin^2\theta\sin\varphi\cos\varphi - 25\cos^2\theta\sin\varphi\cos\varphi}_{-25\sin\varphi\cos\varphi}\big)$$

$$= \langle -25\cos\theta\sin^2\varphi,\ -25\sin\theta\sin^2\varphi,\ -25\sin\varphi\cos\varphi \rangle$$

$$\vec{v} = \langle -y, x, 2z \rangle = \langle -5\sin\theta\sin\varphi,\ 5\cos\theta\sin\varphi,\ 10\cos\varphi \rangle$$

$$\vec{v} \cdot \vec{n} = 125\cos\theta\sin\theta\sin^3\varphi - 125\cos\theta\sin\theta\sin^3\varphi - 250\sin\varphi\cos^2\varphi$$
$$= -250\sin\varphi\cos^2\varphi$$

$$\text{rate of flow} = \int_0^{2\pi}\int_0^{\pi} (1500)(-250\sin\varphi\cos^2\varphi)\, d\varphi\, d\theta$$

$$= (1500)(-250)(2\pi)\int_0^{\pi} \sin\varphi\cos^2\varphi\, d\varphi$$

$u = \cos\varphi$
$du = -\sin\varphi\, d\varphi$
$0 \to 1$
$\pi \to -1$

$$= (250)(1500)(2\pi)\int_1^{-1} u^2\, du$$

$$= 250(1500)(2\pi)\left.\frac{u^3}{3}\right|_1^{-1}$$

$$= \boxed{250(1500)(2\pi)\left(-\frac{2}{3}\right)}$$