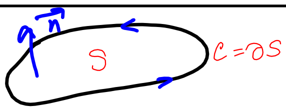
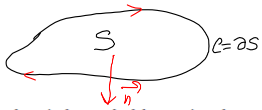




Chapter 14. **Vector calculus.**
Section 14.8 **Stokes' Theorem.**

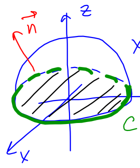
Stokes' Theorem. Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec F$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C \vec F\cdot d\vec r = \iint_S \text{curl}\vec F\cdot d\vec S$$

The Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\vec F$ is equal to the surface integral of the normal component of the curl of $\vec F$.

Example 1. Use Stokes' Theorem to evaluate $\iint_S \text{curl}\vec F\cdot d\vec S$ if $\vec F(x,y,z) = < xyz, x, e^{xy}\cos(z) >$ and S is (the top) hemisphere $x^2+y^2+z^2=1$, oriented upward.

$$\iint_S \text{curl}\,\vec F\cdot d\vec S = \oint_C \vec F\cdot d\vec r = \int_a^b \vec F(\vec r(t))\cdot \vec r\,'(t)\,dt$$



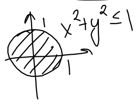
Parametrical equations for C:

$x=\cos t$

$y=\sin t$

$z=0,$

$0\le t\le 2\pi$



$$\vec r(t) = <\underbrace{\cos t}_{x}, \underbrace{\sin t}_{y}, \underbrace{0}_{z}>$$

$$\vec r\,'(t) = <-\sin t, \cos t, 0>$$

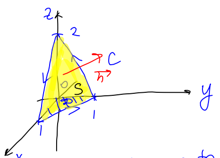
$$\vec F(\vec r(t)) = <0, \cos t, e^{\cos t\sin t}>$$

$$\vec F(\vec r(t))\cdot \vec r\,'(t) = \cos^2 t$$

$$\iint_S \text{curl}\,\vec F\cdot d\vec S = \int_0^{2\pi}\cos^2 t\,dt = \int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt = \left[\frac12 t + \frac14 \overset{0}{\cancel{\sin 2t}}\right]_0^{2\pi} = \boxed{\pi}$$

Example 2. Use Stokes' Theorem to evaluate $\oint_C \vec F\cdot d\vec r$ if $\vec F(x,y,z) = < z^2, y^2, xy >$ and C is the triangle with vertices (1,0,0), (0,1,0), and (0,0,2) and is oriented counterclockwise as viewed from above.

$$\oint_C \vec F\cdot d\vec r = \iint_S \text{curl}\vec F\cdot d\vec S = \iint_D \text{curl}\vec F\cdot \vec n\,dA$$



$S:\ 2x+2y+z=2$

$z = 2-2x-2y$

$\vec n = \pm < z_x, z_y, -1>$

$= - <-2, -2, -1>$

$= <2, 2, 1>$

$$\text{curl}\,\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix}$$

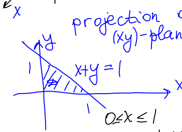
$= <x, -y+2z, 0>$

$\text{curl}\,\vec F\cdot\vec n = 2x - 2y + 4\underbrace{z}_{2-2x-2y}$

$= 2x-2y+4(2-2x-2y)$

$= 8-6x-10y$

projection onto (xy)-plane



$0\le x\le 1$

$0\le y\le 1-x$

$$\oint_C \vec F\cdot d\vec r = \int_0^1\int_0^{1-x}(8-6x-10y)\,dy\,dx$$

$$= \int_0^1 \left(8y - 6xy - 5y^2\right)\Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left(8-8x-6x+6x^2-5\underbrace{(1-2x+x^2)}_{(1-x)^2}\right)dx$$

$$= \int_0^1 (3-4x+x^2)\,dx$$

$$= \left[3x - 2x^2 + \frac{x^3}{3}\right]_0^1$$

$$= \boxed{\frac43}$$