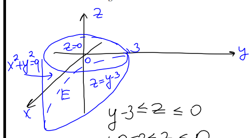
Chapter 14. **Vector calculus.**
Section 14.9 **Divergence Theorem.**

Divergence theorem. Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV}$$

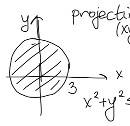
Example 1. Use the Divergence Theorem to calculate the surface integral $\iint_S \vec{F}\cdot d\vec{S}$ if $\vec{F} = < ye^{z^2}, y^2, e^{xy} >$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=y-3$.



$y-3 \le z \le 0$

$r\sin\theta - 3 \le z \le 0$

projection onto (xy)-plane



$x^2+y^2 \le 9$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \cancel{\frac{\partial}{\partial x}(ye^{z^2})}^{\,0} + \frac{\partial}{\partial y}(y^2) + \cancel{\frac{\partial}{\partial z}(e^{xy})}^{\,0}$$

$$= 2y$$

Cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ dV = r\,dz\,dr\,d\theta \end{cases}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^3\int_{r\sin\theta-3}^{0} 2\,\overbrace{r\sin\theta}^{y}\,\overbrace{r\,dz\,dr\,d\theta}^{dV}$$

$$\underbrace{z\Big]_{r\sin\theta-3}^{0} = 0-(r\sin\theta-}$$

$$= 2\int_0^{2\pi}\int_0^3 r^2\sin\theta\,(3-r\sin\theta)\,dr\,d\theta$$

$$= 6\int_0^{2\pi}\int_0^3 r^2\sin\theta\,dr\,d\theta - 2\int_0^{2\pi}\int_0^3 r^3\sin^2\theta\,dr\,d\theta$$

$$= 6\cancel{\int_0^{2\pi}\sin\theta\,d\theta}^{\,0}\int_0^3 r^2\,dr - 2\int_0^3 r^3\,dr\int_0^{2\pi}\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\,d\theta$$

$$= -2\left[\frac{r^4}{4}\right]_{r=0}^{r=3}\left(\frac{\theta}{2} - \cancel{\frac{1}{4}\sin 2\theta}^{\,0}\right)_{\theta=0}^{\theta=2\pi}$$

$$= -2\,\frac{81}{4}\,\frac{2\pi}{2}$$

$$= \boxed{-\frac{81\pi}{2}}$$