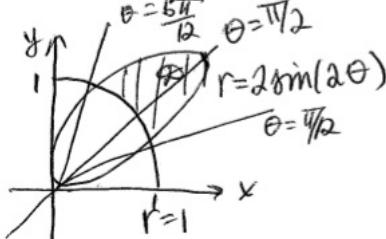


26. Find the area inside one petal of the rose  $r = 2\sin(2\theta)$  outside the circle  $r = 1$ . Sketch the region of integration.



intersection:

$$2\sin 2\theta = 1$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

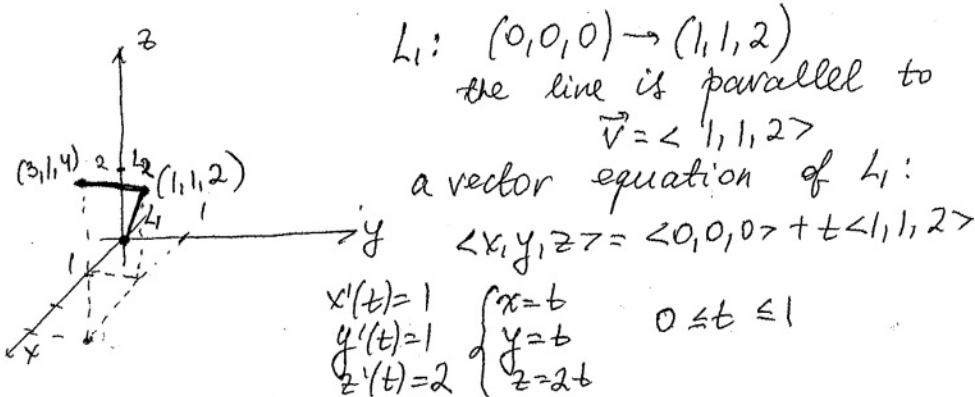
$$\frac{\pi}{12} \leq \theta \leq \frac{5\pi}{12}$$

$$\begin{aligned}
 A &= \iint dA = \int_{\pi/12}^{5\pi/12} \int_{r=1}^{2\sin(2\theta)} r dr d\theta = \int_{\pi/12}^{5\pi/12} \left[ \frac{r^2}{2} \right]_1^{2\sin(2\theta)} d\theta \\
 &= \int_{\pi/12}^{5\pi/12} \left( 2\sin^2 2\theta - \frac{1}{2} \right) d\theta = \int_{\pi/12}^{5\pi/12} [1 - \cos 4\theta] d\theta - \frac{1}{2} \theta \Big|_{\pi/12}^{5\pi/12} \\
 &= \left[ \theta - \frac{1}{4} \sin 4\theta \right] \Big|_{\pi/12}^{5\pi/12} - \frac{1}{2} \theta \Big|_{\pi/12}^{5\pi/12} \\
 &= \frac{1}{2} \left[ \frac{4\pi}{12} - \frac{1}{4} \sin \left[ \frac{20\pi}{12} \right] + \frac{1}{4} \sin \left( \frac{4\pi}{12} \right) \right] \\
 &= \frac{2\pi}{12} - \frac{1}{4} \sin \frac{5\pi}{3} + \frac{1}{4} \sin \frac{\pi}{3} \\
 &= \boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}
 \end{aligned}$$

12. Evaluate the line integral  $\int_C x^3 z ds$  if  $C$  is given by  $x = 2 \sin t$ ,  $y = t$ ,  $z = 2 \cos t$ ,  $0 \leq t \leq \pi/2$ .

$$\begin{aligned}
 \int_C x^3 z ds &= \int_0^{\pi/2} [x'(t)]^3 [z(t)] \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt \\
 &= \int_0^{\pi/2} 8 \sin^3 t \cos t \sqrt{4 \cos^2 t + 1 + 4 \sin^2 t} dt \\
 &= \int_0^{\pi/2} 16 \sin^3 t \cos t \sqrt{5} dt \\
 &= 16\sqrt{5} \int_0^{\pi/2} \sin^3 t \cos t dt = \left| \begin{array}{l} \sin t = u \\ du = \cos t dt \\ t=0 \rightarrow u=\sin 0=0 \\ t=\pi/2 \rightarrow u=\sin \pi/2=1 \end{array} \right| \\
 &= 16\sqrt{5} \int_0^1 u^3 du = 16\sqrt{5} \frac{u^4}{4} \Big|_0^1 = \boxed{4\sqrt{5}}
 \end{aligned}$$

13. Evaluate  $\int_C ydx + zdy + xdz$  if  $C$  consists of the line segments from  $(0,0,0)$  to  $(1,1,2)$  and from  $(1,1,2)$  to  $(3,1,4)$ .



$L_2: (1,1,2) \rightarrow (3,1,4)$   
the line is parallel to  $\vec{v} = \langle 2, 0, 2 \rangle$   
a vector equation of  $L_2$ :  
 $\langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t \langle 2, 0, 2 \rangle$

$x'(t) = 2 \quad \begin{cases} x = 1+2t \\ y = 1+0 \\ z = 2+2t \end{cases} \quad 0 \leq t \leq 1$

$$\begin{aligned} \int_C ydx + zdy + xdz &= \int_{L_1} ydx + zdy + xdz + \int_{L_2} ydx + zdy + xdz \\ &= \int_0^1 [(t)(1) + (2t)(1) + (t)(2)] dt + \int_0^1 [(1)(2) + (2+2t)(0) + (1+2t)/2] dt \\ &= \int_0^1 5t dt + \int_0^1 (4+4t) dt = \left. \frac{5t^2}{2} \right|_0^1 + \left. \left( 4t + \frac{4t^2}{2} \right) \right|_0^1 = \frac{5}{2} + 4 + 2 \\ &= \frac{5}{2} + 6 \\ &= \boxed{\frac{17}{2}} \end{aligned}$$

14. Evaluate  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F}(x, y) = x^2y\vec{i} + e^y\vec{j}$  and  $C$  is given by  $\vec{r}(t) = t^2\vec{i} - t^3\vec{j}$ ,  $0 \leq t \leq 1$ .

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy \quad \text{where}$$

$$P(x, y) = x^2y$$

$$Q(x, y) = e^y$$

$$x(t) = t^2 \quad dx = 2t dt$$

$$y(t) = -t^3 \quad dy = -3t^2 dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left[ (t^2)^2 (-t^3) + e^{-t^3} (-3t^2) \right] dt$$

$$= \int_0^1 (-2t^8) dt + \int_0^1 (-3t^2) e^{-t^3} dt \quad \left. \begin{array}{l} u = -t^3 \\ du = -3t^2 dt \\ t=0 \rightarrow u=0 \\ t=1 \rightarrow u=-1 \end{array} \right\}$$

$$= -2 \frac{t^9}{9} \Big|_0^1 + \int_0^{-1} e^u du$$

$$= -\frac{2}{9} + e^u \Big|_0^{-1} = -\frac{2}{9} (e^{-1} - e^0) = -\frac{2}{9} (e^{-1} - 1)$$

15. Show that  $\vec{F}(x, y) = (2x + y^2 + 3x^2y)\vec{i} + (2xy + x^3 + 3y^2)\vec{j}$  is conservative vector field. Use this fact to evaluate  $\int_C \vec{F} \cdot d\vec{r}$  if  $C$  is the arc of the curve  $y = x \sin x$  from  $(0, 0)$  to  $(\pi, 0)$ .

$$\text{if } P(x, y) = 2x + y^2 + 3x^2y \quad Q(x, y) = 2xy + x^3 + 3y^2 \\ \frac{\partial P}{\partial y} = 2y + 3x^2 \quad \frac{\partial Q}{\partial x} = 2y + 3x^2$$

since  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ , then  $\vec{F}$  is conservative.

$f$  such that  $\nabla f = \vec{F}(x, y)$ :

$$\nabla f = \langle f_x, f_y \rangle$$

$$\begin{cases} f_x = 2x + y^2 + 3x^2y \\ f_y = 2xy + x^3 + 3y^2 \end{cases}$$

$$f(x, y) = \int f_x dx = \int (2x + y^2 + 3x^2y) dx = x^2 + y^2x + x^3y + g(y)$$

$g(y)$  is an unknown function.

plug  $f$  into the 2nd equation:

~~$$f_y = 2xy + x^3 + g'(y) = 2xy + x^3 + 3y^2$$~~

$$g'(y) = 3y^2$$

$$g(y) = y^3 + K, K \text{ is a constant}$$

$$f(x, y) = x^2 + xy^2 + x^3y + y^3 + K$$

$$\int_C \vec{F} \cdot d\vec{r} = f(\pi, 0) - f(0, 0) = \pi^2 + \pi \cdot 0 + \pi^3 \cdot 0 + 0^3 + K - K = \boxed{\pi^2}$$

16. Show that  $\vec{F}(x, y, z) = yz(2x+y)\vec{i} + xz(x+2y)\vec{j} + xy(x+y)\vec{k}$  is conservative vector field. Use this fact to evaluate  $\int_C \vec{F} \cdot d\vec{r}$  if  $C$  is given by  $\vec{r}(t) = (1+t)\vec{i} + (1+2t^2)\vec{j} + (1+3t^3)\vec{k}$ ,  $0 \leq t \leq 1$ .

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2+y^2z & x^2z+2xyz & x^2y+xy^2 \end{vmatrix} = \begin{cases} \vec{F}(0) = \langle 0, 0, 0 \rangle \\ \vec{F}(1) = \langle 2, 3, 4 \rangle \end{cases}$$

$$= \vec{i} \left( \frac{\partial}{\partial y} (x^2y + xy^2) - \frac{\partial}{\partial z} (x^2z + 2xyz) \right)$$

$$- \vec{j} \left( \frac{\partial}{\partial x} (x^2y + xy^2) - \frac{\partial}{\partial z} (2xyz + y^2z) \right)$$

$$+ \vec{k} \left( \frac{\partial}{\partial x} (x^2z + 2xyz) - \frac{\partial}{\partial y} (2xyz + y^2z) \right)$$

$$= \vec{i} (x^2 + 2xy - (x^2 + 2xy)) - \vec{j} (2xy + y^2 - (2xy + y^2)) + \vec{k} (2xz + 2yz - (2xz + 2yz))$$

$\Rightarrow \vec{0} \rightarrow \vec{F}$  is conservative.

$f$  such that  $\nabla f = \langle f_x, f_y, f_z \rangle = \vec{F}$ ;

$$\begin{cases} f_x = 2xyz + y^2z \\ f_y = x^2z + 2xyz \\ f_z = x^2y + xy^2 \end{cases}$$

$$f = \int f_x dx = \int (2xyz + y^2z) dx = x^2yz + xy^2z + g(y, z)$$

$g(y, z)$  is unknown.

Plug  $f$  into the 2nd and the 3rd equations:

$$f_y = x^2z + 2xyz + g_y = x^2z + 2xyz$$

$g_y = 0$ , thus  $g$  does not depend on  $y$ .

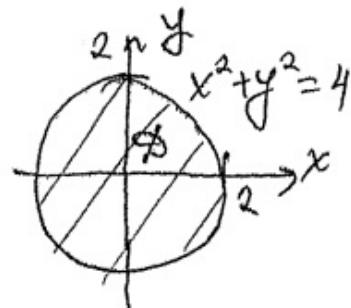
$$f_z = x^2y + xy^2 + g_z = x^2y + xy^2$$

$g_z = 0$ , thus  $g(y, z) = K$  - constant.

$$f(x, y, z) = x^2yz + xy^2z + K$$

$$\int_C \vec{F} \cdot d\vec{r} = f(2, 3, 4) - f(1, 1, 1) = (4)(3)(4) + (2)(9)(4) + K - 1 - 1 - K = \boxed{118}$$

17. Use Green's Theorem to evaluate  $\int_C x^2 y dx - xy^2 dy$  where  $C$  is the circle  $x^2 + y^2 = 4$  with counterclockwise orientation.



$$\begin{aligned} \int_C x^2 y dx - xy^2 dy &= \iint_D \left( \frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2) \right) dA \\ &= \iint_D [-y^2 - x^2] dA = -\iint_D (x^2 + y^2) dA = \end{aligned}$$

polar coordinates!

$$\begin{array}{ll} x = r \cos \theta & dA = r dr d\theta \\ y = r \sin \theta & x^2 + y^2 = r^2 \\ 0 \leq \theta \leq 2\pi & \end{array}$$

$$0 \leq r \leq 2$$

$$\begin{aligned} -\iint_D (x^2 + y^2) dA &= - \int_0^{2\pi} \int_0^2 r^2 r dr d\theta = - \int_0^{2\pi} d\theta \int_0^2 r^3 dr \\ &= -2\pi \frac{r^4}{4} \Big|_0^2 = (-2\pi)(4) \\ &= \boxed{-8\pi} \end{aligned}$$

||      ||      ||  
 18. Find  $\operatorname{curl} \vec{F}$  and  $\operatorname{div} \vec{F}$  if  $\vec{F} = x^2 z \vec{i} + 2x \sin y \vec{j} + 2z \cos y \vec{k}$ .

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(2x \sin y) + \frac{\partial}{\partial z}(2z \cos y)$$

$$= 2xz + 2x \cos y + 2 \cos y$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & 2x \sin y & 2z \cos y \end{vmatrix}$$

$$= \vec{i} \left( \frac{\partial}{\partial y}(2z \cos y) - \frac{\partial}{\partial z}(2x \sin y) \right) - \vec{j} \left( \frac{\partial}{\partial x}(2z \cos y) - \frac{\partial}{\partial z}(x^2 z) \right) + \vec{k} \left( \frac{\partial}{\partial x}(2x \sin y) - \frac{\partial}{\partial y}(x^2 z) \right)$$

$$= -2z \sin y \vec{i} + x^2 \vec{j} + 2 \sin y \vec{k}$$

41. Find an equation of the tangent plane to the surface given by parametric equations  $x = u^2$ ,  $y = u - v^2$ ,  $z = v^2$ , at the point  $(1,0,1)$ .

$$\vec{r}(u,v) = \langle u^2, u-v^2, v^2 \rangle$$

$$\vec{r}_u = \langle 2u, 1, 0 \rangle$$

$$\vec{r}_v = \langle 0, -2v, 2v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 1 & 0 \\ 0 & -2v & 2v \end{vmatrix} = \langle +2v, -4uv, -4v^2 \rangle$$

$$(1,0,1) = \vec{r}(1,1)$$

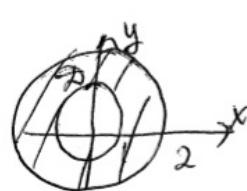
$$[\vec{r}_u \times \vec{r}_v](1,1) = \langle 2, -4, -4 \rangle$$

Tangent plane:

$$\boxed{2(x-1) - 4(y-0) - 4(z-1) = 0}$$

42. Find the area of the hyperbolic paraboloid  $z = x^2 - y^2$  that lies between the cylinders  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

$$A = \iint_S dS = \iint_D |\vec{n}| dA$$



$$z = x^2 - y^2$$

$$\begin{aligned}\vec{n} &= \pm \langle z_x, z_y, -1 \rangle \\ &= \pm \langle 2x, -2y, -1 \rangle\end{aligned}$$

$$|\vec{n}| = \sqrt{1+4x^2+4y^2}.$$

$$A = \iint_D \sqrt{1+4x^2+4y^2} dA$$

Polar coordinates:

$$\begin{aligned}x &= r\cos\theta & 1 \leq r \leq 2 \\ y &= r\sin\theta & 0 \leq \theta \leq 2\pi\end{aligned}$$

$$A = \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2} r dr d\theta = 2\pi \int_1^2 r \sqrt{1+4r^2} dr$$

$$\begin{aligned}1+4r^2 &= u & 1 \rightarrow 5 \\ du &= 8r dr & 2 \rightarrow 17\end{aligned}$$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u} du = \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_5^{17} = \boxed{\frac{\pi}{16} [17^{3/2} - 5^{3/2}]}$$

43. Find the area of the surface with parametric equations  $x = uv$ ,  $y = u+v$ ,  $z = u-v$ ,  $u^2 + v^2 \leq 1$ .

$$A = \iint_S dS = \iint_D |\vec{n}| dA$$

$$\text{where } u^2 + v^2 \leq 1$$

$$\vec{n} = \pm \vec{r}_u \times \vec{r}_v$$

$$\vec{r} = \langle uv, u+v, u-v \rangle$$

$$\vec{r}_u = \langle v, 1, 1 \rangle, \quad \vec{r}_v = \langle u, 1, -1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = \langle -2, u+v, v-u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4 + (u+v)^2 + (v-u)^2}$$

$$= \sqrt{4 + u^2 + 2uv + v^2 + v^2 - 2uv + u^2}$$

$$= \sqrt{4 + 2u^2 + 2v^2}$$

$$A = \iint_D \sqrt{4 + 2u^2 + 2v^2} dA$$

Polar coordinates:  $u = r\cos\theta$        $0 \leq r \leq 1$   
 $v = r\sin\theta$        $0 \leq \theta \leq 2\pi$

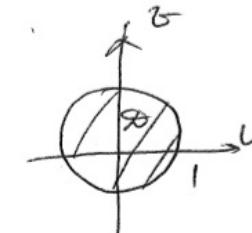
$$A = \int_0^{2\pi} \int_0^1 r \sqrt{4 + 2r^2} dr d\theta = 2\pi \int_0^1 r \sqrt{4 + 2r^2} dr$$

$$u = 4 + 2r^2 \quad \begin{matrix} 1 \rightarrow 6 \\ du = 4rdr \end{matrix}$$

$$dU = 4rdr \quad 0 \rightarrow 4$$

$$= \frac{\pi}{2} \int_4^6 \sqrt{u} du = \frac{\pi}{2} \cdot \frac{2}{3} u^{3/2} \Big|_4^6$$

$$= \boxed{\frac{\pi}{3} [6^{3/2} - 4^{3/2}]}$$



44. Find the mass of a thin funnel in the shape of a cone  $z = \sqrt{x^2 + y^2}$ ,  $1 \leq z \leq 4$  if its density function is  $\rho(x, y, z) = 10 - z$ .

$$m = \iint_S g(x, y, z) dS = \iint_D g(x, y, z(x, y)) |\vec{n}| dA$$

$$\begin{aligned} S: \quad & x = x \\ & y = y \\ & z = \sqrt{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} D: \quad & (\leq \sqrt{x^2 + y^2} \leq 4 \\ & 1 \leq x^2 + y^2 \leq 16 \end{aligned}$$

$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle = \pm \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right\rangle$$

$$|\vec{n}| = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} = \sqrt{2}$$

$$m = \iint_D (10 - \sqrt{x^2 + y^2}) \sqrt{2} dA$$

Polar coordinates:  $x = r \cos \theta$      $1 \leq r \leq 4$   
 $y = r \sin \theta$      $0 \leq \theta \leq 2\pi$

$$= \int_0^{2\pi} \int_1^4 \sqrt{2}(10 - r) r dr d\theta = 2\pi \sqrt{2} \int_1^4 (10r - r^2) dr d\theta$$

$$= 2\sqrt{2}\pi \left( 5r^2 - \frac{r^3}{3} \right) \Big|_1^4$$

$$= 2\sqrt{2}\pi \left( 5(15) - \frac{64}{3} \right)$$

$$= \boxed{108\sqrt{2}\pi}$$

45. Evaluate  $\iint_S yz \, dS$  if  $S$  is the part of the plane  $z = y + 3$  that lies inside the cylinder  $x^2 + y^2 = 1$ .

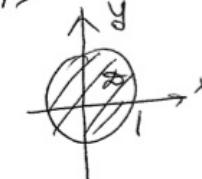
$$\iint_S yz \, dS = \iint_D yz / |\vec{n}| \, dA$$

$$z = y + 3$$

$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle = \pm \langle 0, 1, -1 \rangle$$

$$|\vec{n}| = \sqrt{2}$$

Parameter domain  $D$ :  $x^2 + y^2 \leq 1$



$$\iint_S yz \, dS = \iint_D y(\overbrace{y+3}) \sqrt{2} \, dA$$

$$\left\{ \begin{array}{l} \text{Polar coord.} \\ x = r \cos \theta \quad 0 \leq r \leq 1 \\ y = r \sin \theta \quad 0 \leq \theta \leq 2\pi \end{array} \right.$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 r \cos \theta (r \cos \theta + 3) r dr d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^1 [r^3 \cos^2 \theta + 3r^2 \cos \theta] dr d\theta$$

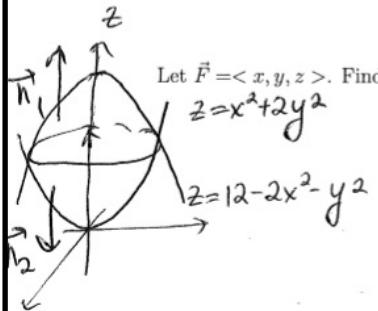
$$= \sqrt{2} \int_0^{2\pi} \left[ \frac{1}{4} \underbrace{\cos^2 \theta}_{1+\cos 2\theta} + \cos \theta \right] d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left[ \frac{1}{8} + \frac{1}{8} \cancel{\cos 2\theta} + \cancel{\cos^2 \theta} \right] d\theta$$

$$= \boxed{\frac{\sqrt{2}\pi}{4}}$$

46. Let  $T$  be the solid bounded by the paraboloids

$$z = x^2 + 2y^2, \text{ and } z = 12 - 2x^2 - y^2.$$



Let  $\vec{F} = \langle x, y, z \rangle$ . Find the outward flux of  $\vec{F}$  across the boundary surface of  $T$ .

$$\text{flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} + \iint_{S_2}$$

$$S_1: z = x^2 + 2y^2$$

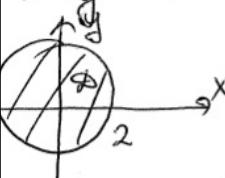
$\vec{n}_2 = \langle 2x, 4y, -1 \rangle$  ( $\vec{n}_2$  is directed downward)

Parameter domain:

$$x^2 + 2y^2 = 12 - 2x^2 - y^2$$

$$x^2 + 3y^2 = 12$$

$$x^2 + y^2 = 4$$



$$= \langle 2x, 4y, -1 \rangle$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n} dA$$

$$= \iint_D \langle x, y, \sqrt{x^2 + 2y^2} \rangle \cdot \langle 2x, 4y, -1 \rangle dA$$

$$= \iint_D (x^2 + 2y^2) dA$$

$$0 \leq r \leq 2$$

$$x = r \cos \theta$$

$$0 \leq \theta \leq 2\pi$$

$$y = r \sin \theta$$

Polar coordinates:

$$= \int_0^{2\pi} \int_0^2 (r^2 \cos^2 \theta + 2r^2 \sin^2 \theta) r dr d\theta = \int_0^{2\pi} \int_0^2 (r^3 + 2r^3 \sin^2 \theta) dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3 (1 + \sin^2 \theta) dr d\theta = \int_0^{2\pi} 4(1 + \frac{1 - \cos 2\theta}{2}) d\theta$$

$$= \int_0^{2\pi} 4 \left( \frac{3}{2} - \frac{\cos 2\theta}{2} \right) d\theta = 6(2\pi) = \boxed{12\pi}$$

$$S_2: z = 12 - 2x^2 - y^2$$

$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle = \pm \langle -4x, -2y, -1 \rangle$$

$\vec{n}$  is directed upward.  
 $= \langle 4x, 2y, 1 \rangle$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n} dA = \iint_D \langle x, y, 12 - 2x^2 - y^2 \rangle \cdot \langle 4x, 2y, 1 \rangle dA$$

$$= \iint_D (2x^2 + y^2 + 12) dA \quad \left| \begin{array}{l} \text{Polar coord.} \\ x = r\cos\theta \\ y = r\sin\theta \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{array} \right|$$

$$= \int_0^{2\pi} \int_0^2 [r^2 + r^2 \cancel{\cos^2\theta} + 12] r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 [r^3(1 + \cos^2\theta) + 12r] dr d\theta$$

$$= \int_0^{2\pi} [4(1 + \cos^2\theta) + 24] d\theta$$

$$= \int_0^{2\pi} [4(1 + \frac{1 + \cos 2\theta}{2}) + 24] d\theta$$

$$= \int_0^{2\pi} [4[\frac{3}{2} + \cancel{\frac{\cos 2\theta}{2}}] + 24] d\theta$$

$$= 30(2\pi) = \boxed{60\pi}$$

$$\iint_S = 60\pi + 12\pi = \boxed{72\pi}$$

'7'

OR Divergence Theorem:

$$\text{flux} = \iiint_E \operatorname{div} \vec{F} dV$$

$$\operatorname{div} \vec{F} = 3,$$

cylindrical coord.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$12 - x^2 - y^2 \geq z \geq x^2 + y^2$$

$$12 - r^2 - r^2 \cos^2 \theta \geq z \geq r^2 + r^2 \sin^2 \theta$$

$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$\text{flux} = 3 \int_0^{2\pi} \int_0^2 \int_{r^2 + r^2 \sin^2 \theta}^{12 - r^2 - r^2 \cos^2 \theta} r dz dr d\theta$$

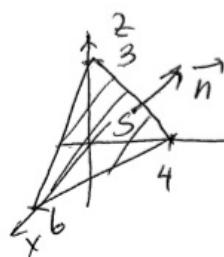
$$= 3 \int_0^{2\pi} \int_0^2 r [12 - r^2 - r^2 \cos^2 \theta - r^2 - r^2 \sin^2 \theta] dr d\theta$$

$$= 3 \int_0^{2\pi} \int_0^2 r [12 - 3r^2] dr d\theta = 9(2\pi) \int_0^2 (4r - r^3) dr$$

$$= 18\pi \left( \frac{4r^2}{2} - \frac{r^4}{4} \right)_0^2$$

$$= 18\pi (16 - 4) = 18\pi(12) = \boxed{72\pi}$$

47. Verify the Divergence Theorem for  $\vec{F} = \langle x^2, xy, z \rangle$  and the region  $E$  bounded by the coordinate planes and the plane  $2x + 3y + 4z = 12$ .



$$\iint_S \vec{F} \cdot d\vec{S} \stackrel{?}{=} \iiint_E \operatorname{div} \vec{F} dV$$

E outward flux.

LHS:  $S_1: 2x+3y+4z=12$

$$z = \frac{1}{4}(12 - 3y - 2x)$$

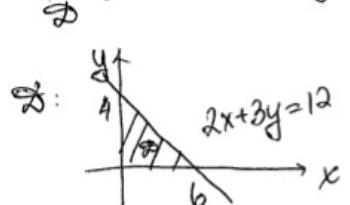
$$\vec{n}_1 = \pm \langle 2x, 2y, -1 \rangle = \pm \left\langle -\frac{1}{2}, -\frac{3}{4}, -1 \right\rangle$$

$$= \left\langle \frac{1}{2}, \frac{3}{4}, 1 \right\rangle$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \left[ \frac{1}{2}x^2 + \frac{3}{4}xy + \underbrace{\frac{1}{4}(12-3y-2x)}_z \right] dA$$

$\vec{F} \cdot \vec{n}$

$$= \iint_D \left[ \frac{1}{2}x^2 + \frac{3}{4}xy + 3 - \frac{3}{4}y - \frac{1}{2}x \right] dA$$



$$0 \leq x \leq 6$$

$$0 \leq y \leq 4 - \frac{2x}{3}$$

$$\begin{aligned} &= \int_0^6 \int_0^{4-\frac{2x}{3}} \left( \frac{1}{2}x^2 + \frac{3}{4}xy + 3 - \frac{3}{4}y - \frac{1}{2}x \right) dy dx \\ &= \int_0^6 \left( \frac{1}{2}x^2y + \frac{3}{4}x \frac{y^2}{2} + 3y - \frac{3}{4} \frac{y^2}{2} - \frac{1}{2}xy \right) \Big|_0^{4-\frac{2x}{3}} dx \\ &= \int_0^6 \left[ \frac{1}{2}x^2 \left( 4 - \frac{2x}{3} \right) + \frac{3}{8}x \left( 4 - \frac{2x}{3} \right)^2 + 3 \left( 4 - \frac{2x}{3} \right) \right] dx \end{aligned}$$

$$\left. - \frac{3}{8} \left( 4 - \frac{2x}{3} \right)^2 - \frac{1}{2}x \left( 4 - \frac{2x}{3} \right) \right] dx = \dots = 66$$

$$\underline{S_2: \quad x=0 \text{ (yz-plane).} \quad \vec{n}_2 = \langle -1, 0, 0 \rangle}$$
$$\vec{F} \cdot \vec{n}_2 = \langle 0, 0, z \rangle \cdot \langle -1, 0, 0 \rangle = 0.$$
$$\iint_{S_2} \vec{F} \cdot d\vec{S} = 0.$$

$$S_3: \quad y=0 \quad (\text{xz-plane.}) \quad \vec{n}_3 = \langle 0, -1, 0 \rangle.$$
$$\vec{F} \cdot \vec{n}_3 = \langle x^2, 0, z \rangle \cdot \langle 0, -1, 0 \rangle = 0$$
$$\iint_{S_3} \vec{F} \cdot d\vec{S} = 0$$

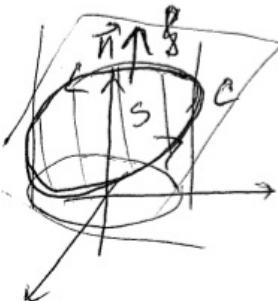
$$\underline{S_4: \quad z=0 \quad (\text{xy-plane}) \quad \vec{n}_4 = \langle 0, 0, -1 \rangle}$$
$$\vec{F} \cdot \vec{n}_4 = \langle x^2, xy, 0 \rangle \cdot \langle 0, 0, -1 \rangle = 0.$$
$$\iint_{S_4} \vec{F} \cdot d\vec{S} = 0$$

$$\overline{\iint_S \vec{F} \cdot d\vec{n} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} = \boxed{66}}$$

RHS:  $\operatorname{div} \vec{F} = 2x + x + 1 = 3x + 1$   
 $0 \leq z \leq 3 - \frac{3}{4}y - \frac{1}{2}x$  |  $\iiint_E \operatorname{div} \vec{F} dV = \int_0^6 \int_0^{4-\frac{2}{3}x} \int_0^{3-\frac{3}{4}y - \frac{1}{2}x} (3x+1) dz dy dx$   
 $0 \leq y \leq 4 - \frac{2}{3}x$   
 $0 \leq x \leq 6$

$$\begin{aligned}
 &= \int_0^6 \int_0^{4-\frac{2}{3}x} (3x+1)(3 - \frac{3}{4}y - \frac{1}{2}x) dy dx \\
 &= \int_0^6 \left( 9x - \frac{9}{4}xy - \frac{3}{2}x^2 + 3 - \frac{3}{4}y - \frac{1}{2}x \right) dy dx \\
 &= \int_0^6 (3x+1) \left( 3y - \frac{3}{8}y^2 - \frac{1}{2}xy \right) \Big|_{y=0}^{y=4-\frac{2}{3}x} dx \\
 &= \int_0^6 (3x+1) \left( 3\left(4 - \frac{2}{3}x\right) - \frac{3}{8}\left(4 - \frac{2}{3}x\right)^2 - \frac{1}{2}x\left(4 - \frac{2}{3}x\right) \right) dx \\
 &= \int_0^6 (3x+1) \left( 6 - 2x + \frac{1}{6}x^2 \right) dx \\
 &= \int_0^6 \left( 16x - 6x^2 + \frac{1}{6}x^3 + 6 \right) dx \\
 &= \left[ 8x^2 - 2x^3 + \frac{x^4}{18} + \frac{x^5}{8} + 6x \right]_0^6 = 8(36) - 2(216) + \frac{1}{18}(216) + \frac{1296}{8} \\
 &\quad + 36 = \boxed{66} . \quad \boxed{LHS = RHS}
 \end{aligned}$$

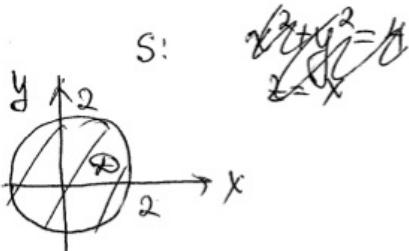
48. Use Stokes Theorem to evaluate  $\int_C \vec{F} \cdot d\vec{r}$  if  $\vec{F} = \langle 2z, x, 3y \rangle$  and C is the ellipse in which the plane  $z = x$  meets the cylinder  $x^2 + y^2 = 4$ , oriented counterclockwise as viewed from above.



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

$$= \iint_S \text{curl } \vec{F} \cdot \vec{n} dA$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & x & 3y \end{vmatrix} = \langle 3, 2, 1 \rangle$$



$$z = x$$

$$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$$

$$= \pm \langle 1, 0, -1 \rangle$$

$$= \langle -1, 0, 1 \rangle$$

Parameter domain:  $\mathcal{D} = \{x^2 + y^2 \leq 4\}$

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} dA = \iint_{\mathcal{D}} (-3+1) dA = -2 \iint_{\mathcal{D}} dA$$

$$= -2(\text{area } \mathcal{D})$$

$$= -2\pi 2^2$$

$$= \boxed{-8\pi}$$