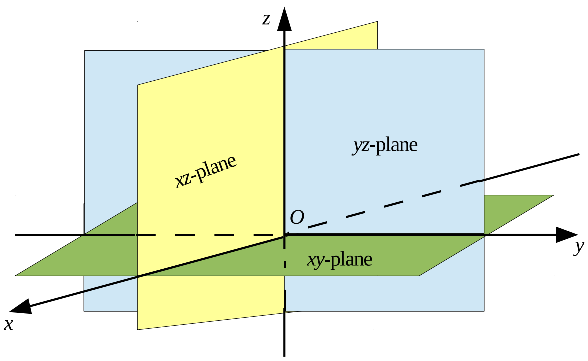
Chapter 12. **Vectors and the geometry of space**
Section 12.1 **Three-dimensional coordinate system**

## 3D Space.

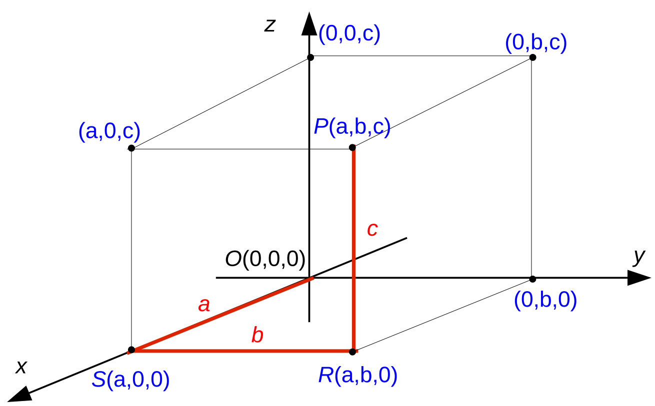
In order to represent points in space, we first choose a fixed point $O$ (the origin) and tree directed lines through $O$ that are perpendicular to each other, called the **coordinate axes** and labeled the $x$-axis, $y$-axis, and $z$-axis. Usually we think of the $x$ and $y$-axes as being horizontal and $z$-axis as being vertical.

The direction of $z$-axis is determined by the **right-hand rule**: if your index finger points in the positive direction of the $x$-axis, middle finger points in the positive direction of the $y$-axis, then your thumb points in the positive direction of the $z$-axis.

The three coordinate axes determine the three **coordinate planes**. The $xy$-plane contains the $x$- and $y$-axes and its equation is $z = 0$, the $xz$-plane contains the $x$- and $z$-axes and its equation is $y = 0$, The $yz$-plane contains the $y$- and $z$-axes and its equation is $x = 0$. These three coordinate planes divide space into eight parts called **octants**. The **first octant** is determined by positive axes.




Take a point $P$ in space, let $a$ be directed distance from $yz$-plane to $P$, $b$ be directed distance from $xz$-plane to $P$, and $c$ be directed distance from $xy$-plane to $P$. We represent the point $P$ by the ordered triple $(a, b, c)$ of real numbers, and we call $a$, $b$, and $c$ the **coordinates** of $P$. The point $P(a, b, c)$ determine a rectangular box. If we drop a perpendicular from $P$ to the $xy$-plane, we get a point $Q(a, b, 0)$ called the **projection** of $P$ on the $xy$-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of $P$ on the $yz$-plane and $xz$-plane, respectively.




The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x, y, z)|x, y, z \in \mathbb{R}\}$ is the set of all ordered triplets of real numbers. We have given a one-to-one correspondence between points $P$ in space and ordered triplets $(a, b, c)$ in $\mathbb{R}^3$. It is called a **tree-dimensional rectangular coordinate system.**

**Example 1.** Sketch the points $(1, 3, 5)$, $(-2, -1, 3)$, $(0, 2, -5)$ on a single set of coordinate axes.

**Example 2.** Draw a rectangular box with the origin and $(2, 4, 5)$ as opposite vertices and with its faces parallel to coordinate planes. Label all vertices of the box.

## Surfaces.

In 3D analytic geometry, an equation in $x$, $y$ and $z$ represents a **surface** in $\mathbb{R}^3$.

**Example 3.** What surfaces in $\mathbb{R}^3$ represented by the following equations?

(a) $z = -6$

(b) $x + y = 1$

(c) $y^2 + z^2 = 1$

## Distance and Spheres.

**The distance formula in three dimensions.** The **distance** $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\boxed{|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\boxed{P_M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)}$$

**Example 4.** Find the length of the sides and medians of a triangle with vertices $A(2, -1, 0)$, $B(4, 1, 1)$, $C(4, -5, 4)$.

**Equation of a sphere** of radius $R$ and center $C(a, b, c)$ is

$$\boxed{(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2}$$

**Example 5.** Find an equation of a sphere that has center $C(-1, 2, 4)$ and passes through the point $(-1, 1, -2)$.

**Example 6.** Show that the equation

$$x^2 + y^2 + z^2 + 8x - 6y + 2z + 17 = 0$$

represents a sphere. Find its center and radius.