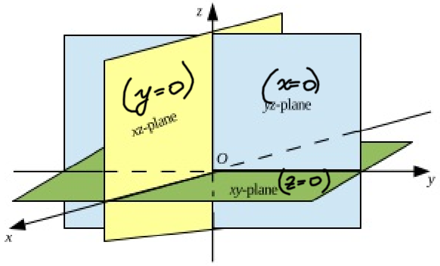
3D Space.

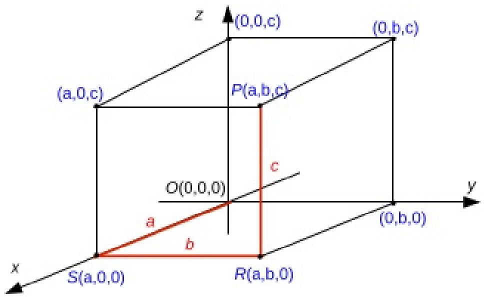
In order to represent points in space, we first choose a fixed point O (the origin) and tree directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x and y-axes as being horizontal and z-axis as being vertical.

The direction of z-axis is determined by the **right-hand rule**: if your index finger points in the positive direction of the x-axis, middle finger points in the positive direction of the y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes**. The xy-plane contains the x- and y-axes and its equation is $z = 0$, the xz-plane contains the x- and z-axes and its equation is $y = 0$, The yz-plane contains the y- and z-axes and its equation is $x = 0$. These three coordinate planes divide space into eight parts called **octants**. The **first octant** is determined by positive axes.



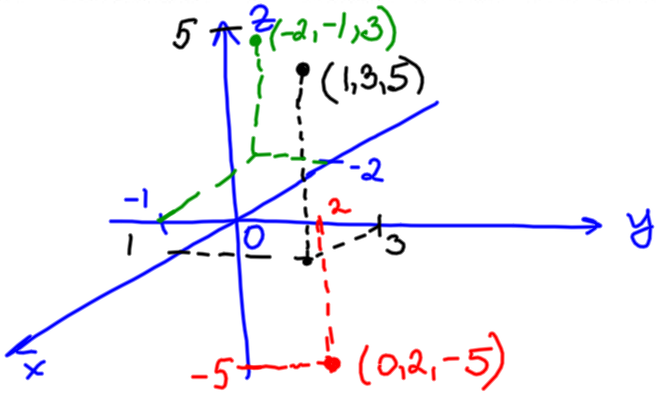
Take a point P in space, let a be directed distance from yz-plane to P, b be directed distance from xz-plane to P, and c be directed distance from xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers, and we call a, b, and c the **coordinates** of P. The point $P(a, b, c)$ determine a rectangular box. If we drop a perpendicular from P to the xy-plane, we get a point $Q(a, b, 0)$ called the **projection** of P on the xy-plane. Similarly, $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P on the yz-plane and xz-plane, respectively.



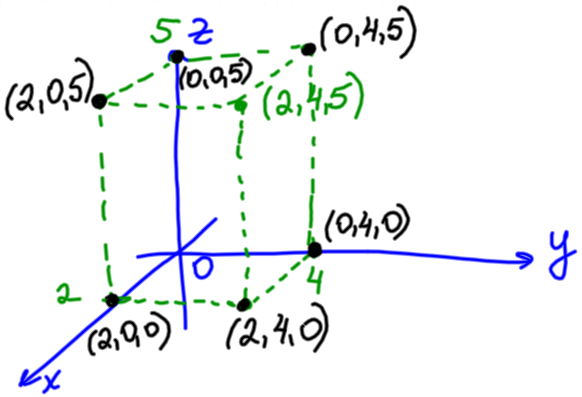
The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is the set of all ordered triplets of real numbers. We have given a one-to-one correspondence between points P in space and ordered triplets (a, b, c) in $\mathbb{R}^3$. It is called a **tree-dimensional rectangular coordinate system.**

Example 1. Sketch the points $(1, 3, 5)$, $(-2, -1, 3)$, $\underbrace{(0, 2, -5)}$ on a single set of coordinate axes.

lies in the yz-plane



Example 2. Draw a rectangular box with the origin and $(2, 4, 5)$ as opposite vertices and with its faces parallel to coordinate planes. Label all vertices of the box.

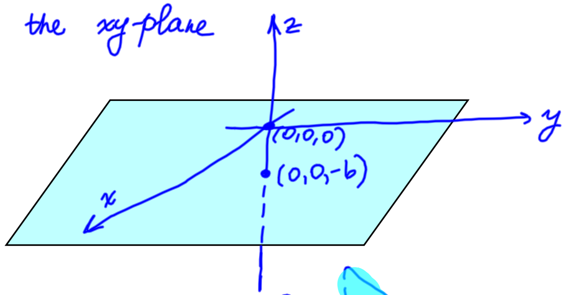


Surfaces.

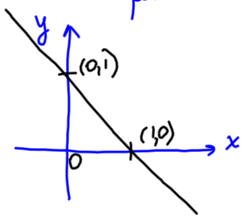
In 3D analytic geometry, an equation in x, y and z represents a **surface** in $\mathbb{R}^3$.

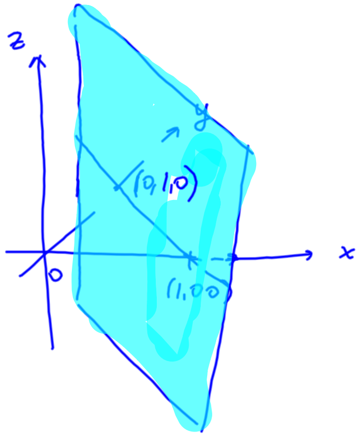
Example 3. What surfaces in $\mathbb{R}^3$ represented by the following equations?

(a) $z = -6$ – a plane parallel to the xy-plane

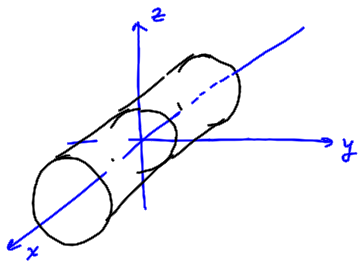


(b) $x + y = 1$ a plane parallel to the z-axis





(c) $y^2 + z^2 = 1$ – cylinder (circular) parallel to the x-axis



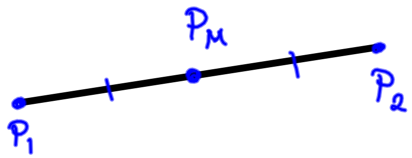
Distance and Spheres.

The distance formula in three dimensions. The **distance** $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

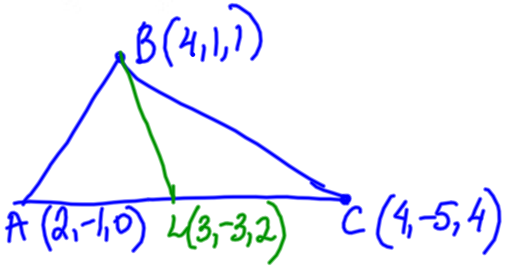
$$\boxed{|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\boxed{P_M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)}$$



Example 4. Find the length of the sides and medians of a triangle with vertices $A(2, -1, 0)$, $B(4, 1, 1)$, $C(4, -5, 4)$.



$|AC| = \sqrt{(4-2)^2 + (-5-(-1))^2 + (4-0)^2} = \sqrt{4+16+16} = \sqrt{36} = \boxed{6}$

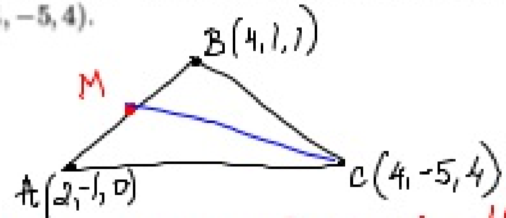
L is the midpoint of the segment AC.

$L\left(\frac{2+4}{2}, \frac{-1-5}{2}, \frac{4+0}{2}\right)$

$L(3, -3, 2)$

$|BL| = \sqrt{(3-4)^2 + (-3-1)^2 + (2-1)^2} = \sqrt{1+16+1} = \sqrt{18} = \boxed{3\sqrt{2}}$

Example 4. Find the length of the sides and medians of a triangle with vertices $A(2,-1,0)$, $B(4,1,1)$, $C(4,-5,4)$.



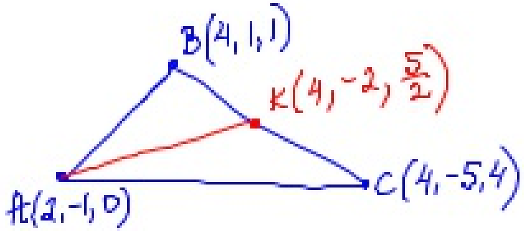
$$|AB| = \sqrt{(4-2)^2+(1-(-1))^2+(1-0)^2} = \sqrt{4+4+1} = \boxed{3}$$

M is the midpoint of the segment AB

$$M\left(\frac{2+4}{2}, \frac{1-1}{2}, \frac{1+0}{2}\right)$$

$$M(3, 0, 1/2)$$

$$|CM| = \sqrt{(3-4)^2+(0-(-5))^2+(1/2-4)^2} = \sqrt{1+25+\frac{49}{4}} = \sqrt{\frac{153}{4}} = \boxed{\frac{\sqrt{153}}{2}}$$



$$|BC| = \sqrt{(4-4)^2+(-5-1)^2+(4-1)^2} = \sqrt{36+9} = \sqrt{45} = \boxed{3\sqrt{5}}$$

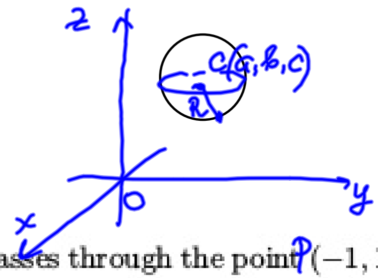
K is the midpoint of the segment BC

$$K\left(\frac{4+4}{2}, \frac{1-5}{2}, \frac{1+4}{2}\right)$$

$$K\left(4, -2, \frac{5}{2}\right)$$

$$|AK| = \sqrt{(4-2)^2+(-2-(-1))^2+\left(\frac{5}{2}\right)^2} = \sqrt{4+1+\frac{25}{4}} = \sqrt{\frac{45}{4}} = \boxed{\frac{3\sqrt{5}}{2}}$$

Equation of a sphere of radius R and center $C(a,b,c)$ is



$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2=R^2}$$

Example 5. Find an equation of a sphere that has center $C(\underbrace{-1,2,4}_{a\ b\ c})$ and passes through the point $P(-1,1,-2)$.

$$R^2 = |CP|^2 = (-1-(-1))^2+(1-2)^2+(-2-4)^2 = 1+36=37$$

$$\boxed{(x+1)^2+(y-2)^2+(z-4)^2=37}$$

Example 6. Show that the equation

$$x^2+y^2+z^2+8x-6y+2z+17=0$$

represents a sphere. Find its center and radius.

$$(x^2+8x)+(y^2-6y)+(z^2+2z)+17=0$$

$$(x^2+8x+16)+(y^2-6y+9)+(z^2+2z+1)-\cancel{16}-9-\cancel{1}+\cancel{17}=0$$

$$(x+4)^2+(y-3)^2+(z+1)^2=9$$

$$\boxed{R=3,\ \text{center } @\ (-4,3,-1)}$$