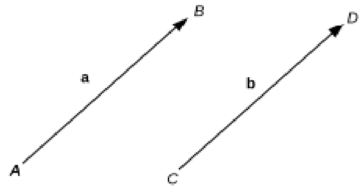
Section 12.2 **Vectors.**

The term **vector** indicates a quantity that has both direction and magnitude. A vector is represented by an arrow or a directed line segment.



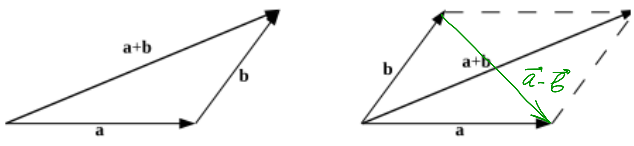
If a particle moves from a point A to a point B, then the corresponding displacement vector **a** has **initial point** A and **terminal point** B and we write $\mathbf{a} = \overrightarrow{AB}$. The directed line segment $\overrightarrow{AB}$ is a **representation** of the vector **a**.

Two vectors that have the same length and direction are called **equivalent** or **equal** even if they have different initial and terminal points. Since $|AB| = |CD|$ adb both segments have the same direction, $\mathbf{a} = \mathbf{b}$.

The only vector with length 0 is the **zero vector 0**. This vector is the only vector with no specific direction.

Vector addition.

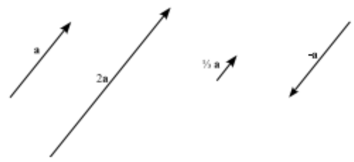
If two vectors **a** and **b** positioned so the initial point of **b** is at the terminal point of **a**, then the **sum** $\mathbf{a} + \mathbf{b}$ is the vector from the initial point of **a** to the terminal point of **b**.



Triangle Law   Parallelogram Law

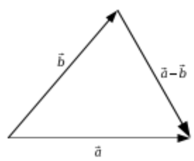
Scalar multiplication.

If c is a scalar and **a** is a vector, then $c\mathbf{a}$ is a vector whose length is $|c|$ times the length of **a** and has the same direction as **a** ($c > 0$) or the opposite direction to **a** ($c < 0$). If either $c = 0$ or $\mathbf{a} = \mathbf{0}$, then $c\mathbf{a} = \mathbf{0}$.



Two vectors **a** and **b** are called **parallel** if $\mathbf{b} = c\mathbf{a}$ for some scalar c.

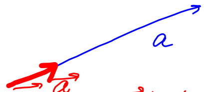
Vector difference. $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b}$



The magnitude (length) $|\mathbf{a}|$ of $\mathbf{a}$ is the length of any its representation.

A unit vector is a vector whose length is 1.

A vector $\boxed{\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a}}$ is a unit vector in the direction of $\mathbf{a}$.



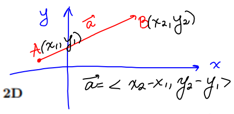
$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$, $|\vec{u}| = 1$

Properties of vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and k and m are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
6. $(k + m)\mathbf{a} = k\mathbf{a} + m\mathbf{a}$
7. $(km)\mathbf{a} = k(m\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Components.



$\vec{a} = < x_2 - x_1, y_2 - y_1 >$

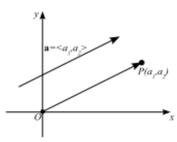
$\vec{a} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 >$ in 3D.

2D

A **two-dimensional vector** is an ordered pair $\mathbf{a} =< a_1, a_2 >$ of real numbers. The numbers a_1 and a_2 are called the **components** of $\mathbf{a}$.

A **representation** of the vector $\mathbf{a} =< a_1, a_2 >$ is a directed line segment $\overrightarrow{AB}$ from any point $A(x, y)$ to the point $B(x + a_1, y + a_2)$.

A particular representation of $\mathbf{a} =< a_1, a_2 >$ is the directed line segment $\overrightarrow{OP}$ from the origin to the point $P(a_1, a_2)$, and $\mathbf{a} =< a_1, a_2 >$ is called the **position vector** of the point $P(a_1, a_2)$.



If $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$\boxed{\overrightarrow{AB} =< x_2 - x_1, y_2 - y_1 >}$$

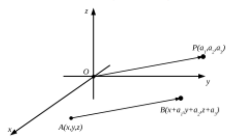
$$\boxed{|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}}$$

3D

A **tree-dimensional vector** is an ordered triple $\mathbf{a} =< a_1, a_2, a_3 >$ of real numbers. The numbers a_1, a_2, and a_3 are called the **components** of $\mathbf{a}$.

A **representation** of the vector $\mathbf{a} =< a_1, a_2, a_3 >$ is a directed line segment $\overrightarrow{AB}$ from any point $A(x, y, z)$ to the point $B(x + a_1, y + a_2, z + a_3)$.

A particular representation of $\mathbf{a} =< a_1, a_2, a_3 >$ is the directed line segment $\overrightarrow{OP}$ from the origin to the point $P(a_1, a_2, a_3)$, and $\mathbf{a} =< a_1, a_2, a_3 >$ is called the **position vector** of the point $P(a_1, a_2, a_3)$.



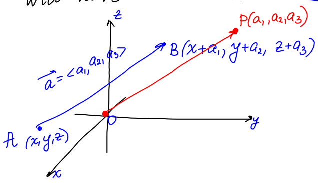
If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then

$$\boxed{\overrightarrow{AB} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >}$$

$$\boxed{|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

2

To draw a vector $\vec{a} = < a_1, a_2, a_3 >$ you pick a random initial point $A(x, y, z)$, then the terminal point B will have the coordinates $B(x + a_1, y + a_2, z + a_3)$



$\overrightarrow{OP}$ is the representation of $\vec{a}$ that starts at the origin.

$\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$

2D

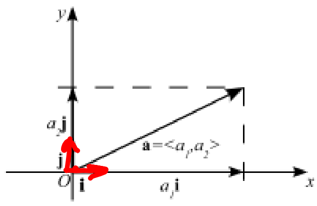
$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$

$c\mathbf{a} = \langle ca_1, ca_2 \rangle$

$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

$\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$., $|\vec{i}| = |\vec{j}| = 1$

$\boxed{\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}}$



$\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

3D

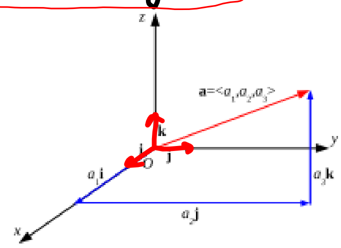
$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

$c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

$\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$, $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

$\boxed{\mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}}$



$|2\vec{a} - 5\vec{b}| \neq 2|\vec{a}| - 5|\vec{b}|$

Example 1. Find $|2\mathbf{a} - 5\mathbf{b}|$ if

1. $\mathbf{a} = \langle -1, 2 \rangle$, $\mathbf{b} = \langle -2, -1 \rangle$

$2\vec{a} - 5\vec{b} = 2\underbrace{\langle -1, 2 \rangle}_{\vec{a}} - 5\underbrace{\langle -2, -1 \rangle}_{\vec{b}}$

$= \langle -2, 4 \rangle - \langle -10, -5 \rangle$

$= \langle -2 - (-10), 4 - (-5) \rangle$

$= \langle 8, 9 \rangle$

$|\langle 8, 9 \rangle| = \sqrt{8^2 + 9^2} = \sqrt{64 + 81} = \boxed{\sqrt{145}}$

2. $\mathbf{a} = \langle 3, -1, 2 \rangle$, $\mathbf{b} = \langle 4, 2, -1 \rangle$

$2\vec{a} - 5\vec{b} = 2\langle 3, -1, 2 \rangle - 5\langle 4, 2, -1 \rangle$

$= \langle 6, -2, 4 \rangle - \langle 20, 10, -5 \rangle$

$= \langle -14, -12, 9 \rangle$

$|\langle -14, -12, 9 \rangle| = \sqrt{(-14)^2 + (-12)^2 + 9^2} = \sqrt{196 + 144 + 81} = \boxed{\sqrt{421}}$

Find a vector **a** with representation given by the directed line segment $\overrightarrow{AB}$.

$A(0, 2, 2), \quad B(2, 2, -3)$

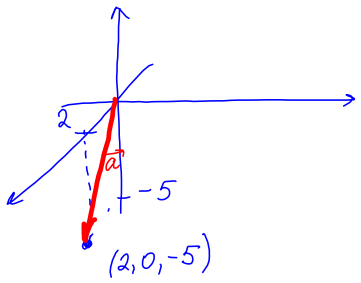
$\langle 2, 0, -5 \rangle$

$\overrightarrow{AB} = \langle 2-0,\ 2-2,\ -3-2 \rangle = \boxed{\langle 2, 0, -5 \rangle}$

B

A

Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.



z

Example 2. Find the unit vector in the direction of the given vector.

1. $\mathbf{a} = 5\mathbf{i} - 3\mathbf{j} = \langle 5, -3 \rangle, \quad |\langle 5, -3 \rangle| = \sqrt{25+9} = \sqrt{34}$

$\vec{u} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\langle 5, -3 \rangle}{\sqrt{34}} = \boxed{\left\langle \dfrac{5}{\sqrt{34}}, -\dfrac{3}{\sqrt{34}} \right\rangle}$

~~2. $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.~~

Find the vector that has the same direction as $\langle 2, 9, -6 \rangle$ but has length 3.

$\left\langle \frac{6}{11}, \frac{27}{11}, -\frac{18}{11} \right\rangle$

$|\langle 2,9,-6 \rangle| = \sqrt{2^2+9^2+(-6)^2} = \sqrt{4+81+36} = \sqrt{121} = 11$

The unit vector in the direction of $\langle 2,9,-6 \rangle$

$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2,9,-6 \rangle}{11} = \langle \frac{2}{11}, \frac{9}{11}, -\frac{6}{11} \rangle, \ |\vec{u}| = 1$

$3\vec{u} = 3\langle \frac{2}{11}, \frac{9}{11}, -\frac{6}{11} \rangle = \boxed{\langle \frac{6}{11}, \frac{27}{11}, -\frac{18}{11} \rangle}$