Section 12.3 **The dot product.**

Definition. The **dot** or **scalar product** of two nonzero vectors $\vec{a}$ and $\vec{b}$ is the number

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, we define $\vec{a} \cdot \vec{b} = 0$.

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, b_3 >$, then

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and

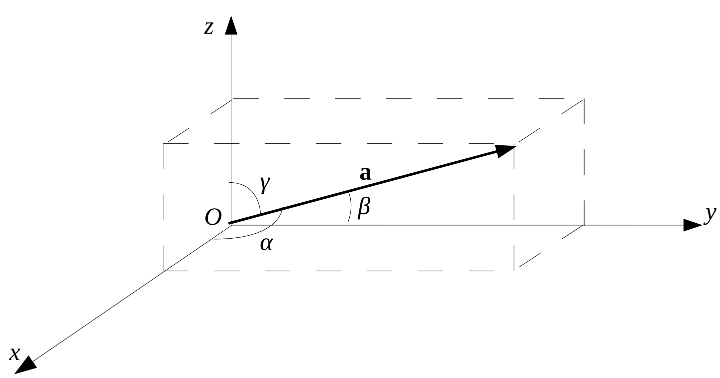
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\pi/2$.

Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

Example 1. Find the values of x such that the vectors $\vec{a} =< x, 1, 2 >$ and $\vec{b} =< 3, 4, x >$ are orthogonal.

Direction angles and direction cosines. The **direction angles** of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\vec{a}$.



$$\boxed{\cos\alpha = \frac{\vec{a} \cdot \vec{\imath}}{|\vec{a}||\vec{\imath}|} = \frac{a_1}{|\vec{a}|}}, \boxed{\cos\beta = \frac{a_2}{|\vec{a}|}}, \boxed{\cos\gamma = \frac{a_3}{|\vec{a}|}}.$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = 1$$

We can write

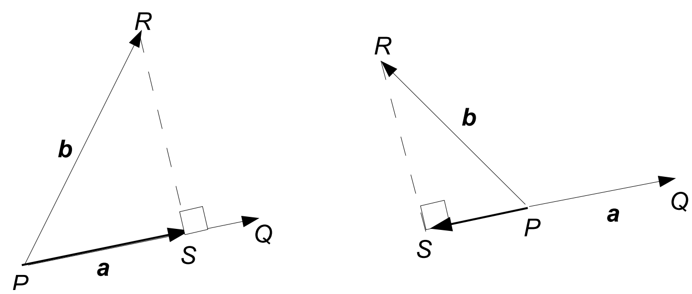
$$\vec{a} = < a_1, a_2, a_3 > = < |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma > = |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} = < \cos\alpha, \cos\beta, \cos\gamma >$$

which says that the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$.

Example 2. Find the direction cosines of the vector $< -4, -1, 2 >$.



$\vec{PS} = \text{proj}_{\vec{a}}\vec{b}$ is called the **vector projection of $\vec{b}$ onto $\vec{a}$**.

$|\vec{PS}| = \text{comp}_{\vec{a}}\vec{b}$ is called the **scalar projection of $\vec{b}$ onto $\vec{a}$** or the **component of $\vec{b}$ along $\vec{a}$**. The scalar projection of $\vec{b}$ onto $\vec{a}$ is the length of the vector projection of $\vec{b}$ onto $\vec{a}$ if $0 \le \theta < \pi/2$ and is negative if $\pi/2 \le \theta < \pi$.

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}} \quad \boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} < a_1, a_2, a_3 >}$$

Example 3. Find the scalar and vector projections of $\vec{b} = < 4, 2, 0 >$ onto $\vec{a} = < 1, 2, 3 >$.