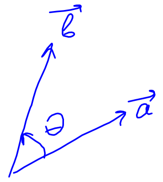
Section 12.3 **The dot product.**



Definition. The dot or scalar product of two nonzero vectors $\vec a$ and $\vec b$ is the number

$$\boxed{\vec a \cdot \vec b = |\vec a||\vec b|\cos\theta}$$

where θ is the angle between $\vec a$ and $\vec b$, $0 \le \theta \le \pi$. If either $\vec a$ or $\vec b$ is $\vec 0$, we define $\vec a \cdot \vec b = 0$.

If $\vec a = \langle a_1, a_2, a_3 \rangle$ and $\vec b = \langle b_1, b_2, b_3 \rangle$, then

$$\boxed{\vec a \cdot \vec b = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

and

$$\boxed{\cos\theta = \frac{\vec a \cdot \vec b}{|\vec a||\vec b|}}$$

$\cos\frac{\pi}{2} = 0.$

Two nonzero vectors $\vec a$ and $\vec b$ are called perpendicular or orthogonal if the angle between them is $\pi/2$.

Two vectors $\vec a$ and $\vec b$ are orthogonal if and only if $\vec a \cdot \vec b = 0$.

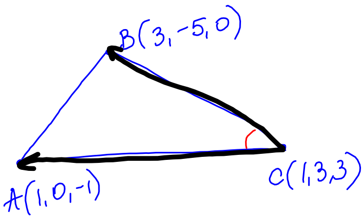
Example 1. Find the values of x such that the vectors $\vec a = \langle x, 1, 2 \rangle$ and $\vec b = \langle 3, 4, x \rangle$ are orthogonal.

$\vec a \cdot \vec b = \langle x, 1, 2 \rangle \cdot \langle 3, 4, x \rangle = 3x + 4(1) + 2x = 5x + 4 = 0$

$$\boxed{x = -4/5}$$

Find, correct to the nearest degree, the three angles of the triangle with the given vertices.

$A(1, 0, -1)$, $B(3, -5, 0)$, $C(1, 3, 3)$



$\cos(\angle ACB)$

Take two vectors that start @ C:

$\vec{CA} = \langle 1-1, -3+0, -3+(-1) \rangle = \langle 0, -3, -4 \rangle$

$\vec{CB} = \langle 3-1, -5-3, 0-3 \rangle = \langle 2, -8, -3 \rangle$

$$\cos(\angle C) = \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}| \cdot |\vec{CB}|} = \frac{\langle 0, -3, -4 \rangle \cdot \langle 2, -8, -3 \rangle}{\sqrt{3^2+4^2}\sqrt{2^2+8^2+3^2}}$$

$$= \frac{0(2) + 3(8) + 3(4)}{5\sqrt{77}} = \frac{36}{5\sqrt{77}}$$

$$\angle C = \arccos\left(\frac{36}{5\sqrt{77}}\right)$$

Direction angles and direction cosines. The direction angles of a nonzero vector $\vec{a}$ are the angles α, β, and γ in the interval $[0, \pi]$ that $\vec{a}$ makes with the positive $x-$, $y-$, and $z-$ axes. The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the direction cosines of the vector $\vec{a}$.

$\vec{a} = <a_1, a_2, a_3>$

$\vec{i} = <1, 0, 0>$

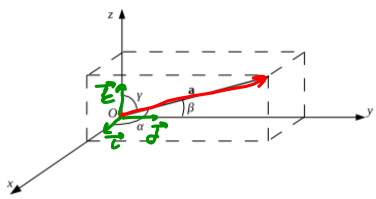
$\vec{j} = <0, 1, 0>$

$\vec{k} = <0, 0, 1>$

α is the angle between $\vec{a}$ and $\vec{i}$

$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}|}$

$= \frac{a_1 + 0 + 0}{|\vec{a}|} = \frac{a_1}{|\vec{a}|}$



$\boxed{\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|}}$ $\boxed{\cos\beta = \frac{a_2}{|\vec{a}|}}$ $\boxed{\cos\gamma = \frac{a_3}{|\vec{a}|}}$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a_1}{|\vec{a}|}\right)^2 + \left(\frac{a_2}{|\vec{a}|}\right)^2 + \left(\frac{a_3}{|\vec{a}|}\right)^2 = 1$$

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We can write

$\vec{a} = <a_1, a_2, a_3> = |\vec{a}| < \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} > = |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$

$$\vec{a} = < a_1, a_2, a_3 > = < |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma > = |\vec{a}| < \cos\alpha, \cos\beta, \cos\gamma >$$

Therefore

$$\frac{1}{|\vec{a}|}\vec{a} = < \cos\alpha, \cos\beta, \cos\gamma >$$

which says that the direction cosines of $\vec{a}$ are the components of the unit vector in the direction of $\vec{a}$.

Example 2. Find the direction cosines of the vector $< -4, -1, 2 >$.

$|< -4, -1, 2 >| = \sqrt{16 + 1 + 4} = \sqrt{21}$

$\cos\alpha = -\frac{4}{\sqrt{21}}, \quad \cos\beta = \frac{-1}{\sqrt{21}}, \quad \cos\gamma = \frac{2}{\sqrt{21}}$

Vector and scalar projections.

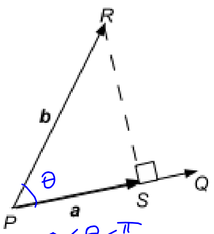
$\triangle PRS$ right triangle

$|\vec{PS}| = |\vec{PR}|\cos\theta$

$= \cancel{|\vec{b}|}\cdot\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cancel{|\vec{b}|}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

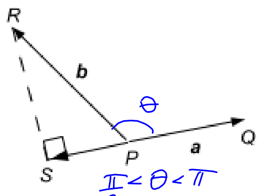
$\vec{PS} = (\text{comp}_{\vec{a}}\vec{b})\frac{\vec{a}}{|\vec{a}|}$

unit vector in the direction of $\vec{a}$



$0 < \theta < \frac{\pi}{2}$

$\text{comp}_{\vec{a}}\vec{b} = -|\vec{PS}|$



$\frac{\pi}{2} < \theta < \pi$

$\vec{PS} = \text{proj}_{\vec{a}}\vec{b}$ is called the **vector projection of $\vec{b}$ onto $\vec{a}$.**

$|\vec{PS}| = \text{comp}_{\vec{a}}\vec{b}$ is called the **scalar projection of $\vec{b}$ onto $\vec{a}$** or the **component of $\vec{b}$ along** $\vec{a}$. The scalar projection of $\vec{b}$ onto $\vec{a}$ is the length of the vector projection of $\vec{b}$ onto $\vec{a}$ if $0 \le \theta < \pi/2$ and is negative if $\pi/2 \le \theta < \pi$.

$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$ $\boxed{\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} < a_1, a_2, a_3 >}$

Example 3. Find the scalar and vector projections of $\vec{b} = < 4, 2, 0 >$ onto $\vec{a} = < 1, 2, 3 >$.

$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{<4,2,0>\cdot<1,2,3>}{|<1,2,3>|} = \frac{4(1)+2(2)+3(0)}{\sqrt{1+4+9}} = \boxed{\frac{8}{\sqrt{14}}}$

$\text{proj}_{\vec{a}}\vec{b} = (\text{comp}_{\vec{a}}\vec{b})\frac{\vec{a}}{|\vec{a}|} = \frac{8}{\sqrt{14}}\cdot\frac{<1,2,3>}{\sqrt{14}} = \frac{8}{14}<1,2,3> = \boxed{<\frac{4}{7}, \frac{8}{7}, \frac{12}{7}>}$