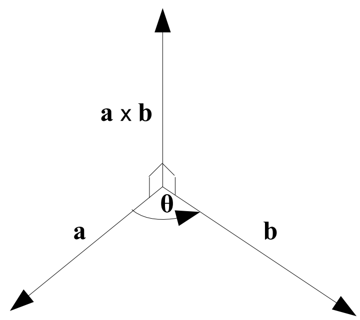
Section 12.4 **The cross product.**

Definition. If $\mathbf{a}$ and $\mathbf{b}$ are nonzero three-dimensional vectors, the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a}\times\mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\mathbf{n}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the **right-hand rule:** If the fingers of your hand curl through the angle θ from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{n}$.



If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then we define $\mathbf{a}\times\mathbf{b}$ to be $\mathbf{0}$.

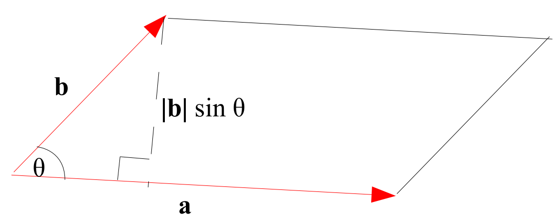
$\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

Properties of the cross product. If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and k is a scalar, then

1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $(k\mathbf{a})\times\mathbf{b} = k(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(k\mathbf{b})$
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
4. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$

The length of the cross product $\mathbf{a}\times\mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.



The cross product in component form.
The cross product of a $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$$(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example 1. If $\mathbf{a} =< -2, 3, 4 >$ and $\mathbf{b} =< 3, 0, 1 >$, find $\mathbf{a} \times \mathbf{b}$.

Example 2. Find the area of the triangle with vertices $A(1, 2, 3)$, $B(2, -1, 1)$, $C(0, 1, -1)$.

Example 3. Find a unit vector orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Triple products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product** of the vectors **a**, **b**, and **c**.

The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Suppose that **a**, **b**, and **c** are given in component form:

$$\mathbf{a} =< a_1, a_2, a_3 >, \quad \mathbf{b} =< b_1, b_2, b_3 >, \quad \mathbf{c} =< c_1, c_2, c_3 > .$$

Then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 4. Find the volume of the parallelepiped determined by vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$.

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is called the **vector triple product** of the vectors **a**, **b**, and **c**.

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$