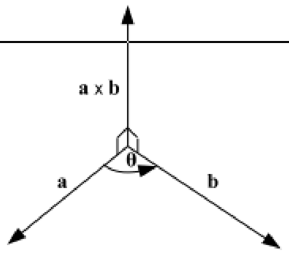
HW over 12.3, 12.4, 12.5 is due 2/1, 11:55 PM

Section 12.4 **The cross product.**

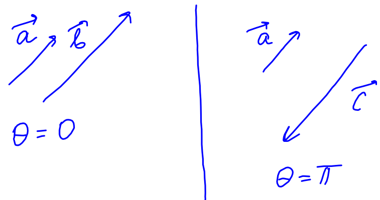
Definition. If **a** and **b** are nonzero three-dimensional vectors, the cross product of **a** and **b** is the vector

$$\boxed{\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin \theta)\mathbf{n}}$$

where θ is the angle between **a** and **b** and **n** is a unit vector perpendicular to both **a** and **b** and whose direction is given by the right-hand rule: If the fingers of your hand curl through the angle θ from **a** to **b**, then your thumb points in the direction of **n**.



Parallel vectors



$\sin 0 = \sin \pi = 0$

If either **a** or **b** is **0**, then we define $\mathbf{a} \times \mathbf{b}$ to be **0**.

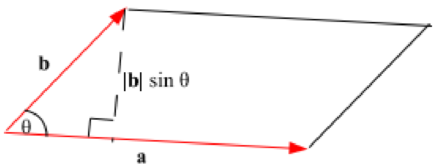
$\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

Two nonzero vectors **a** and **b** are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Properties of the cross product. If **a**, **b**, and **c** are vectors and k is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**.



The cross product in component form.
The cross product of a $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$$\boxed{\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}}$$

$$\left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right| = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

Example 1. If $\mathbf{a} =< -2, 3, 4 >$ and $\mathbf{b} =< 3, 0, 1 >$, find $\mathbf{a} \times \mathbf{b}$.

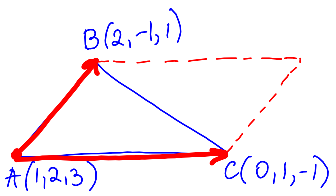
$$\vec{a} \times \vec{b} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -2 & 3 & 4 & -2 & 3 \\ 3 & 0 & 1 & 3 & 0 \end{array}\right|$$

$$= \vec{i}(3)(1) + \vec{j}(4)(3) + \vec{k}(-2)(0) - \vec{k}(3)(3) - \vec{i}(4)(0) - \vec{j}(-2)(1)$$

$$= 3\vec{i} + 12\vec{j} - 9\vec{k} + 2\vec{j}$$

$$= \boxed{3\vec{i} + 14\vec{j} - 9\vec{k}}$$

Example 2. Find the area of the triangle with vertices $A(1, 2, 3)$, $B(2, -1, 1)$, $C(0, 1, -1)$.



$$A_{\Delta} = \frac{1}{2} (\text{area of the parallelogram})$$

$$A_{\Delta} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

$$\vec{AB} = <2-1, -1-2, 1-3> = < 1, -3, -2>$$

$$\vec{AC} = < 0-1, 1-2, -1-3> = < -1, -1, -4>$$

$$\vec{AB} \times \vec{AC} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & -3 & -2 & 1 & -3 \\ -1 & -1 & -4 & -1 & -1 \end{array}\right|$$

$$= 12\vec{i} + 2\vec{j} - \vec{k} - 3\vec{k} - 2\vec{i} + 4\vec{j}$$

$$= 10\vec{i} + 6\vec{j} - 4\vec{k}$$

$$\left|\vec{AB} \times \vec{AC}\right| = \sqrt{10^2 + 6^2 + 4^2} = \sqrt{100 + 36 + 16} = \sqrt{152}$$

$$A_{\Delta} = \frac{\sqrt{152}}{2} = \boxed{\sqrt{38}}$$

Example 3. Find ~~a unit~~ two vectors orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}+\mathbf{k}$.

$\vec{a} = \vec{i} + \vec{j} = \langle 1, 1, 0 \rangle$

$\vec{b} = \vec{i} - \vec{j} + \vec{k} = \langle 1, -1, 1 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 1 & 1 \\ 1 & -1 \end{matrix} = \vec{i} - \vec{k} - \vec{k} - \vec{j} = \boxed{\vec{i} - \vec{j} - 2\vec{k}}$$

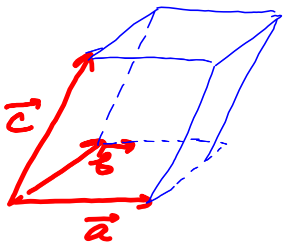
$$-(\vec{a} \times \vec{b}) = \boxed{-\vec{i} + \vec{j} + 2\vec{k}}$$

Triple products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the scalar triple product of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the ~~magnitude~~ absolute value of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$



$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Suppose that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are given in component form:

$$\mathbf{a} = < a_1, a_2, a_3 >, \quad \mathbf{b} = < b_1, b_2, b_3 >, \quad \mathbf{c} = < c_1, c_2, c_3 > .$$

Then

$$\boxed{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

Example 4. Find the volume of the parallelepiped determined by vectors $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$, and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{k}$.

$\vec{a} = \langle 2, 3, -2 \rangle$, $\vec{b} = \langle 1, -1, 0 \rangle$, $\vec{c} = \langle 2, 0, 3 \rangle$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} \begin{matrix} 2 & 3 \\ 1 & -1 \\ 2 & 0 \end{matrix} = -6 + 0 + 0 - 9 + 0 - 4 = -19$$

$$\boxed{V = 19}$$

Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar (lie in the same plane) if and only if $\boxed{\vec{a} \cdot (\vec{b} \times \vec{c}) = 0}$

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is called the **vector triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

$$\boxed{\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \underbrace{(\mathbf{a} \cdot \mathbf{c})}_{\text{dot product}}\mathbf{b} - \underbrace{(\mathbf{a} \cdot \mathbf{b})}_{\text{dot product}}\mathbf{c}.}$$