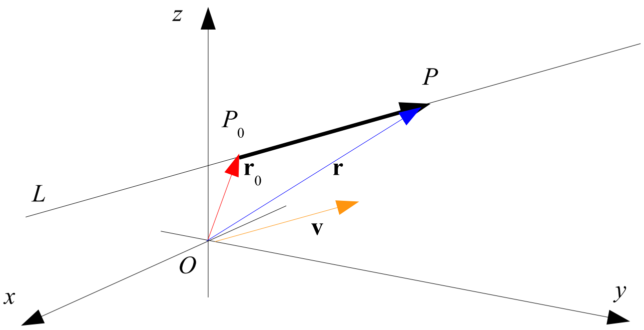
Section 12.5 **Equations of lines and planes.**

A line $L$ in 3D space is determined when we know a point $P_0(x_0, y_0, z_0)$ on $L$ and the direction of $L$. Let $\mathbf{v}$ be a vector parallel to $L$, $P(x, y, z)$ be an arbitrary point on $L$ and $\mathbf{r}_0$ and $\mathbf{r}$ be position vectors of $P_0$ and $P$.




$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{P_0P}$. Since $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, there is a scalar $t$ such that $\overrightarrow{P_0P} = t\mathbf{v}$. Thus a **vector equation** of $L$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Each value of the **parameter** $t$ gives the position vector $\mathbf{r}$ of a point on $L$.

If $\mathbf{r}_0 =< x_0, y_0, z_0 >$, $\mathbf{r} =< x, y, z >$, and $\mathbf{v} =< a, b, c >$, then

$$< x, y, z >=< x_0 + ta, y_0 + tb, z_0 + tc >$$

or

$$x = x_0 + ta, \qquad y = y_0 + tb, \qquad z = z_0 + tc$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line $L$ through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} =< a, b, c >$.

If a vector $\mathbf{v} =< a, b, c >$ is used to describe the direction of a line $L$, then the numbers $a$, $b$, and $c$ are called **direction numbers** of $L$.

**Example 1.** Find the vector equation and parametric equations for the line passing through the point $P(1, -1, -2)$ and parallel to the vector $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Since vectors $\mathbf{v} =< a, b, c >$ and $\overrightarrow{P_0P} =< x - x_0, y - y_0, z - z_0 >$ are parallel, then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of $L$. If one of $a$, $b$, or $c$ is 0, we can still write symmetric equations. For instance, if $c = 0$, then the symmetric equations of $L$ are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0.$$

This means that $L$ lies in the plane $z = z_0$.

**Example 2.** Find symmetric equations for the line passing through the given points:

(a) $(2, -6, 1)$, $(1, 0, -2)$

(b) $(-1, 2, -4)$, $(-1, -3, 2)$

**Example 3.** Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, and $z = 5 - 7t$.

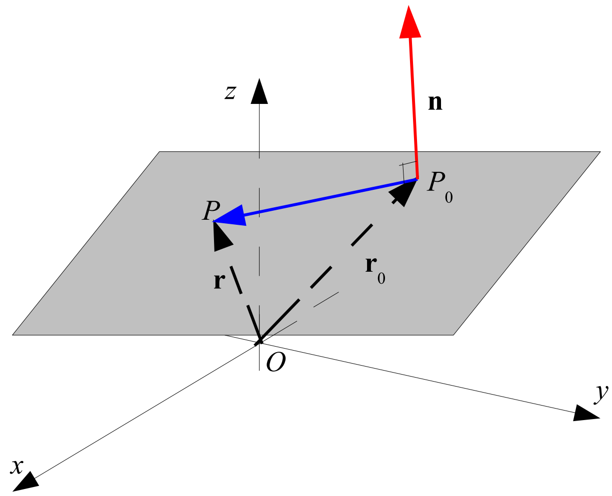
**Example 4.** Determine whether the lines $L_1$ and $L_2$ are parallel, skew (do not intersect and are not parallel), or intersecting. If they intersect, find the point of intersection.

(a) $L_1: \dfrac{x-4}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-3}$, $L_2: \dfrac{x-2}{1} = \dfrac{y+1}{3} = \dfrac{z}{3}$

(b) $L_1: \dfrac{x-1}{2} = \dfrac{y}{1} = \dfrac{z-1}{4}$, $L_2: \dfrac{x}{1} = \dfrac{y+2}{2} = \dfrac{z+2}{3}$

(c) $L_1: x = -6t, y = 1 + 9t, z = -3t,$
$L_2: x = 1 + 2s, y = 4 - 3s, z = s.$

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. $\mathbf{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane and $\mathbf{r}$ and $\mathbf{r}_0$ be the position vectors of $P$ and $P_0$.




$\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$. The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane, in particular, $\mathbf{n}$ is orthogonal to $\overrightarrow{P_0P}$.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

Either of the two equations is called a **vector equation of the plane**.

Let $\mathbf{n} =< a, b, c >$, $\mathbf{r}_0 =< x_0, y_0, z_0 >$, and $\mathbf{r} =< x, y, z >$, then

$$\mathbf{v} \cdot (\mathbf{r} - \mathbf{r}_0) = a(x - x_0) + b(y - y_0) + c(z - z_0)$$

so we can rewrite the vector equation in the following way:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This equation is called the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\mathbf{n} =< a, b, c >$.

By collecting terms in a scalar equation of a plane, we can rewrite the equation as

$$ax + by + cz + d = 0,$$

where $d = -ax_0 - by_0 - cz_0$.

**Example 5.** Find the equation of the plane through the point $P_0(-5, 1, 2)$ with the normal vector $\mathbf{n} =< 3, -3, -1 >$.

**Example 6.** Find the equation of the plane passing through the points $(-1, 1, -1)$, $(1, -1, 2)$, $(4, 0, 3)$.

**Example 7.** Find an equation of the plane that passes through the point $(1, 6, -4)$ and contains the line $x = 1 + 2t$, $y = 2 - 3t$, $z = 3 - t$.

Two planes are **parallel** if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

**Example 8.** (a) Find the angle between the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

(b) Find symmetric equation for the line of intersection of the planes.

**Problem.** Find a formula for the distance $D$ from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

**Example 9.** Find the distance between the parallel planes $x + 2y - z = -1$ and $3x + 6y - 3z = 4$.