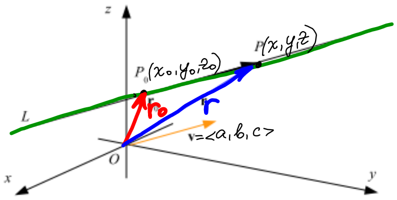
Section 12.5 **Equations of lines and planes.**

A line L in 3D space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. Let $\mathbf{v}$ be a vector parallel to L, $P(x, y, z)$ be an arbitrary point on L and $\mathbf{r}_0$ and $\mathbf{r}$ be position vectors of P_0 and P.



Triangle Law

$\vec{r}_0 + \overrightarrow{P_0P} = \vec{r}$

$\overrightarrow{P_0P} = < x-x_0, y-y_0, z-z_0 >$

$\overrightarrow{P_0P}$ and $\vec{v}$ are parallel

There is a number t such that $\overrightarrow{P_0P} = t\vec{v}$

$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{P_0P}$. Since $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$, there is a scalar t such that $\overrightarrow{P_0P} = t\mathbf{v}$. Thus a vector equation of L is

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.} \qquad \boxed{<x,y,z> = <x_0,y_0,z_0> + t<a,b,c>}$$

Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L.

If $\mathbf{r}_0 =< x_0, y_0, z_0 >$, $\mathbf{r} =< x, y, z >$, and $\mathbf{v} =< a, b, c >$, then

$<x,y,z> = <x_0+ta,\ y_0+tb,\ z_0+tc>$

$$< x, y, z > = < x_0 + ta, y_0 + tb, z_0 + tc >$$

or

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$

$$\boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \quad \text{parametric equations.}}$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} =< a, b, c >$.

If a vector $\mathbf{v} =< a, b, c >$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

Example 1. Find the vector equation and parametric equations for the line passing through the point $P(\underbrace{1}_{x_0}, \underbrace{-1}_{y_0}, \underbrace{-2}_{z_0})$ and parallel to the vector $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. $= <\underbrace{3}_{a}, \underbrace{-2}_{b}, \underbrace{1}_{c}>$

vector equation: $\boxed{<x,y,z> = <1,-1,-2> + t<3,-2,1>}$

$<x,y,z> = <1+3t,\ -1-2t,\ -2+t>$

$$\boxed{\begin{cases} x = 1+3t \\ y = -1-2t \\ z = -2+t \end{cases}} \quad \text{parametric equations}$$

Parametric equations: $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$ $\qquad t = \frac{x-x_0}{a}$, $t = \frac{y-y_0}{b}$, $t = \frac{z-z_0}{c}$

Since vectors $\mathbf{v} =< a, b, c >$ and $\overrightarrow{P_0P} =< x - x_0, y - y_0, z - z_0 >$ are parallel, then

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}} \quad \text{symmetric equations.}$$

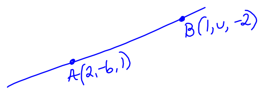
These equations are called **symmetric equations** of L. If one of a, b, or c is 0, we can still write symmetric equations. For instance, if $c = 0$, then the symmetric equations of L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0.$$

This means that L lies in the plane $z = z_0$.

Example 2. Find symmetric equations for the line passing through the given points:

(a) $A(2, -6, 1)$, $B(1, 0, -2)$



$\overrightarrow{AB} = <1-2, 0-(-6), -2-1>$

$= <-1, 6, -3>$

A is a point on the line:

$\frac{x-2}{-1} = \frac{y-(-6)}{6} = \frac{z-1}{-3}$

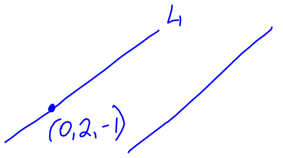
$$\boxed{\frac{x-2}{-1} = \frac{y+6}{6} = \frac{z-1}{-3}}$$

(b) $C(-1, 2, -4)$, $D(-1, -3, 2)$

$\overrightarrow{CD} = <0, -5, 6>$ parallel to the line.

$$\boxed{x = -1 \quad \frac{y-2}{-5} = \frac{z+4}{6}}$$

Example 3. Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, and $z = 5 - 7t$.



$x = 1+2t$
$y = 3t$
$z = 5-7t$

$\vec{v} = \langle 2, 3, -7 \rangle$ is parallel to both lines

$$\boxed{\frac{x-0}{2} = \frac{y-2}{3} = \frac{z+1}{-7}}$$

or

$$\boxed{\frac{x}{2} = \frac{y-2}{3} = \frac{z+1}{-7}}$$

Example 4. Determine whether the lines L_1 and L_2 are parallel, skew (do not intersect and are not parallel), or intersecting. If they intersect, find the point of intersection.

(a) $L_1: \dfrac{x-4}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-3}$, $L_2: \dfrac{x-2}{1} = \dfrac{y+1}{3} = \dfrac{z}{3}$

$\vec{v}_1 = \langle 2, 4, -3 \rangle$, $\vec{v}_2 = \langle 1, 3, 3 \rangle$

$\vec{v}_1$ and $\vec{v}_2$ are not parallel.

Point of intersection.

Write down parametric equations for L_1 and L_2

$L_1: \dfrac{x-4}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-3} = t$

$x = 4+2t$
$y = -5+4t$
$z = 1-3t$

$L_2: \dfrac{x-2}{1} = \dfrac{y+1}{3} = \dfrac{z}{3} = s$

$x = 2+s$
$y = -1+3s$
$z = 3s$

$$\begin{cases} 4+2t = 2+s \\ -5+4t = -1+3s \\ 1-3t = 3s \end{cases}$$

$s = 2+2t$

$1-3t = 3(2+2t)$
$1-3t = 6+6t$
$t = -\frac{5}{9}$, $s = 2+2\left(-\frac{5}{9}\right) = \frac{8}{9}$

plug $t = -\frac{5}{9}$ and $s = \frac{8}{9}$ into the 2nd equation:

$-5+4\left(-\frac{5}{9}\right)$ $\qquad$ $-1+3\left(\frac{8}{9}\right)$

$-\frac{65}{9} \neq \frac{15}{9}$

$\boxed{\text{skew lines}}$

[redacted]

(b) $L_1 : \frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{4}$, $L_2 : \frac{x}{1} = \frac{y+2}{2} = \frac{z+2}{3}$

$\vec{v}_1 = \langle 2, 1, 4 \rangle$, $\vec{v}_2 = \langle 1, 2, 3 \rangle$ not parallel

Point of intersection:

Parametric equations of L_1 and L_2

$\frac{x-1}{2} = \frac{y}{1} = \frac{z-1}{4} = t$ $\qquad \frac{x}{1} = \frac{y+2}{2} = \frac{z+2}{3} = s$

$\begin{cases} x = 1+2t \\ y = t \\ z = 1+4t \end{cases}$ $\qquad \begin{cases} x = s \\ y = -2+2s \\ z = -2+3s \end{cases}$

$\begin{cases} 1+2t = s \\ t = -2+2s \\ 1+4t = -2+3s \end{cases} \longrightarrow t = -2+2(1+2t)$

$t = 0+4t \Rightarrow t=0 \Rightarrow s=1$

Plug $t=0$ and $s=1$ into the 3rd equation:

$1+4(0) \qquad -2+3(1)$

$1 = 1$

$\boxed{\text{intersecting}}$

Point of intersection $(1, 0, 1)$

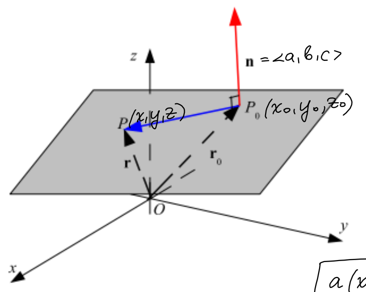
(c) $L_1 :\ x = -6t, y = 1 + 9t, z = -3t,$
$L_2 :\ x = 1 + 2s, y = 4 - 3s, z = s.$

$\vec{v}_1 = \langle -6, 9, -3 \rangle$
$\vec{v}_2 = \langle 2, -3, 1 \rangle$

$\boxed{\text{parallel}}$

$\vec{v}_1 = -3\vec{v}_2$

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. **$\mathbf{n}$ is called a normal vector.** Let $P(x, y, z)$ be an arbitrary point in the plane and $\mathbf{r}$ and $\mathbf{r}_0$ be the position vectors of P and P_0.



$P(x,y,z)$ is an arbitrary point in the plane.
$\vec{P_0P}$ and $\vec{n}$ are orthogonal, then
$\vec{P_0P}\cdot\vec{n}=0$
$\vec{P_0P} = <x-x_0, y-y_0, z-z_0>$
$<x-x_0, y-y_0, z-z_0>\cdot<a,b,c>=0$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

$\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$. The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane, in particular, $\mathbf{n}$ is orthogonal to $\overrightarrow{P_0P}$.

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0 \quad \text{or} \quad \mathbf{n}\cdot\mathbf{r}=\mathbf{n}\cdot\mathbf{r}_0$$

Either of the two equations is called a **vector equation of the plane.**

Let $\mathbf{n} =< a, b, c >$, $\mathbf{r}_0 =< x_0, y_0, z_0 >$, and $\mathbf{r} =< x, y, z >$, then

$$\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}_0) = a(x-x_0)+b(y-y_0)+c(z-z_0)$$

so we can rewrite the vector equation in the following way:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

This equation is called the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\mathbf{n} =< a, b, c >$.

By collecting terms in a scalar equation of a plane, we can rewrite the equation as

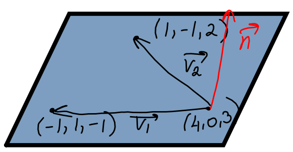
$$ax+by+cz+d=0,$$

where $d = -ax_0 - by_0 - cz_0$.

Example 5. Find the equation of the plane through the point $P_0(\overset{x_0}{-5}, \underset{y_0}{1}, \overset{z_0}{2})$ with the normal vector $\mathbf{n} =< \underset{a}{3}, \underset{b}{-3}, \underset{c}{-1} >$.

$$\boxed{3(x+5)-3(y-1)-1(z-2)=0}$$

Example 6. Find the equation of the plane passing through the points $(-1, 1, -1)$, $(1, -1, 2)$, $(4, 0, 3)$.



$\vec{V_1} = <-5, 1, -4>$

$\vec{V_2} = <-3, -1, -1>$

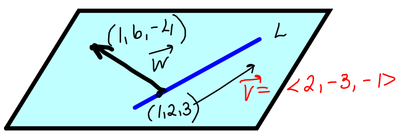
$$\vec{n} = \vec{V_1}\times\vec{V_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & -4 \\ -3 & -1 & -1 \end{vmatrix} = \vec{i}(-1-4) - \vec{j}(5-12) + \vec{k}(5+3)$$

$$= -5\vec{i} - 7\vec{j} + 8\vec{k}$$

$-5(x+1) - 7(y-1) + 8(z+1) = 0$ or $-5(x-1) - 7(y+1) + 8(z-2) = 0$ or $-5(x-4) - 7y + 8(z-3) = 0$

$$\boxed{-5x - 7y + 8z - 4 = 0}$$

Example 7. Find an equation of the plane that passes through the point $(1, 6, -4)$ and contains the line $x = 1 + 2t$, $y = 2 - 3t$, $z = 3 - t$.



$\vec{W} = <0, 4, -7>$

$\vec{V} = <2, -3, -1>$

$$\vec{n} = \vec{W}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 4 & -7 \\ 2 & -3 & -1 \end{vmatrix}$$

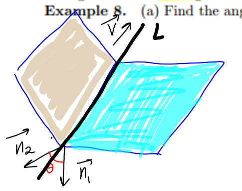
$$= \vec{i}(-4-21) - \vec{j}(0+14) + \vec{k}(0-8)$$

$$= -25\vec{i} - 14\vec{j} - 8\vec{k}$$

$$\boxed{+25(x-1) + 14(y-6) + 8(z+4) = 0}$$

Two planes are parallel if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.

Example 8. (a) Find the angle between the planes $x - 2y + z = 1$ and $2x + y + z = 1$.



$\vec{n}_1 = <1,-2,1>$ $\vec{n}_2 = <2,1,1>$ not parallel

$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{<1,-2,1>\cdot<2,1,1>}{\sqrt{1+4+1}\cdot\sqrt{4+1+1}} = \frac{2-2+1}{6} = \frac{1}{6}$

$\theta = \cos^{-1}\left(\frac{1}{6}\right) \approx \boxed{80^\circ}$

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(b) Find symmetric equation for the line of intersection of the planes.

$\vec{V} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \vec{i}(-2-1) - \vec{j}(1-2) + \vec{k}(1+4)$

$= -3\vec{i} + \vec{j} + 5\vec{k} = <-3,1,5>$

vector parallel to the line.

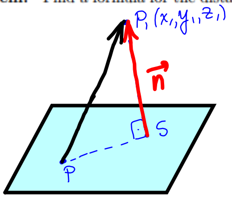
Point on the line $\begin{cases} x-2y+z=1 \\ 2x+y+z=1 \end{cases}$ $x=0$

$\begin{cases} -2y+z=1 \\ y+z=1 \end{cases} \Rightarrow y=0,\ z=1$

$(0,0,1)$

symmetric equations : $\boxed{\frac{x-0}{-3} = \frac{y-0}{1} = \frac{z-1}{5}}$

Problem. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.



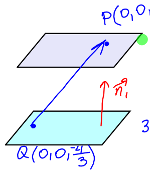
Pick a point P in the plane

$D = |P_1S| = |\text{comp}_{\vec{n}} \vec{PP_1}|$

$\vec{n} = <a,b,c>$

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Example 9. Find the distance between the parallel planes $x + 2y - z = -1$ and $3x + 6y - 3z = 4$.



$x+2y-z=-1$, $\vec{n}_1 = <1,2,-1>$

$3x+6y-3z=4$, $\vec{n}_2 = <3,6,-3>$ $\Rightarrow$ $\vec{n}_2 = 3\vec{n}_1$ $\Rightarrow \vec{n}_1$ and $\vec{n}_2$ are parallel

$\vec{PQ} = <0,0,\frac{7}{3}>$

$D = |\text{comp}_{\vec{n}_1} \vec{PQ}| = \left|\frac{<0,0,\frac{7}{3}>\cdot<1,2,-1>}{\sqrt{1+4+1}}\right| = \boxed{\frac{7}{3\sqrt{6}}}$