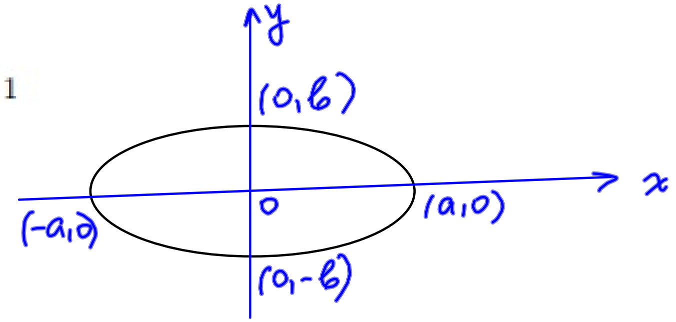
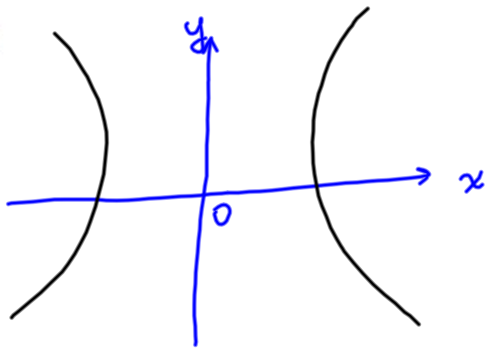
Section 12.6 **Cylinders and quadric surfaces.**

Curves in $\mathbb{R}^2$:

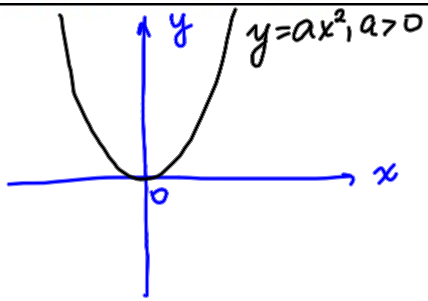
ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

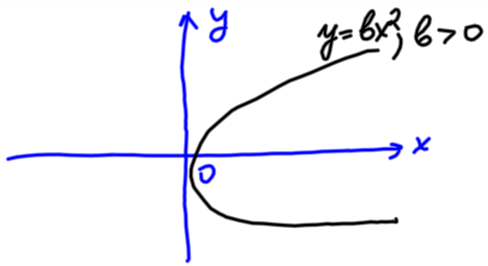


hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



parabola $y=ax^2$ or $x=by^2$





A **quadric surface** is the graph of a second degree equation in three variables. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, ..., J$ are constants. By translation and rotation the equation can be brought into one of two standard forms

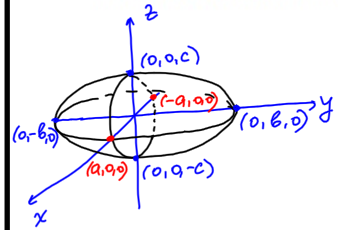
$$Ax^2 + By^2 + Cz^2 + J = 0 \text{ or } Ax^2 + By^2 + Iz = 0$$

In order to sketch the graph of a quadric surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called traces of the surface.

Ellipsoids. The quadric surface with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is called an ellipsoid because all of its traces are ellipses.



Traces: $x=k$: $\frac{k^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{\frac{y^2}{b^2}+\frac{z^2}{c^2}}{1-\frac{k^2}{a^2}}=\frac{1-\frac{k^2}{a^2}}{1-\frac{k^2}{a^2}}$$

$$\frac{y^2}{b^2\left(1-\frac{k^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{k^2}{a^2}\right)}=1 \quad \text{ellipse}$$

$y=k$: $\frac{x^2}{a^2\left(1-\frac{k^2}{b^2}\right)}+\frac{z^2}{c^2\left(1-\frac{k^2}{b^2}\right)}=1$ ellipse

$z=k$: $\frac{x^2}{a^2\left(1-\frac{k^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{k^2}{c^2}\right)}=1$ ellipse

The six intercepts of the ellipsoid are $(\pm a,0,0)$, $(0,\pm b,0)$, and $(0,0,\pm c)$ and the ellipsoid lies in the box $|x|\le a$, $|y|\le b$, $|z|\le c$

Since the ellipsoid involves only even powers of x, y, and z, the ellipsoid is symmetric with respect to each coordinate plane.

Example 1. Find the traces of the surface

$$\frac{4x^2+9y^2+36z^2}{36}=\frac{36}{36} \quad \text{— ellipsoid}$$

in the planes $x=k$, $y=k$, and $z=k$. Identify the surface and sketch it.

$$\frac{4x^2}{36}+\frac{9y^2}{36}+\frac{36z^2}{36}=1$$

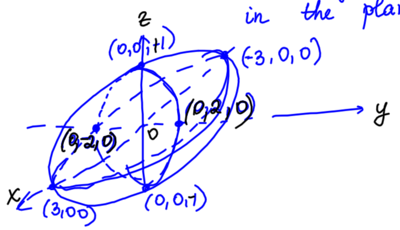
$$\frac{x^2}{9}+\frac{y^2}{4}+z^2=1$$

Traces $x=k$: $\frac{y^2}{4}+z^2=1-\frac{k^2}{9}$

$$\frac{y^2}{4\left(1-\frac{k^2}{9}\right)}+\frac{z^2}{1-\frac{k^2}{9}}=1 \quad \text{ellipse}$$

similarly, we'll get ellipses in the planes $y=k$ and $z=k$.

$\boxed{\text{Ellipsoid}}$



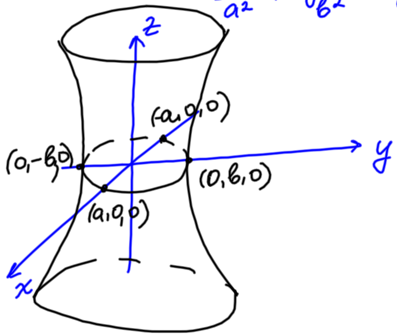
Hyperboloids.

- **Hyperboloid of one sheet.** The quadric surface with equations

  1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
  2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
  3. $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

  are called hyperbolids of one sheet since two of its traces are hyperbolas and one is an ellipse, and its surface consists of just one piece.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



Traces: $x=k$: $\frac{y^2}{b^2\left(1-\frac{k^2}{a^2}\right)} - \frac{z^2}{c^2\left(1-\frac{k^2}{a^2}\right)} = 1$ hyperbola

$y=k$: $\frac{x^2}{a^2\left(1-\frac{k^2}{b^2}\right)} - \frac{z^2}{c^2\left(1-\frac{k^2}{b^2}\right)} = 1$ hyperbola

$z=k$: $\frac{x^2}{a^2\left(1+\frac{k^2}{c^2}\right)} + \frac{y^2}{b^2\left(1+\frac{k^2}{c^2}\right)} = 1$ ellipse

Example 2. Find the traces of the surface

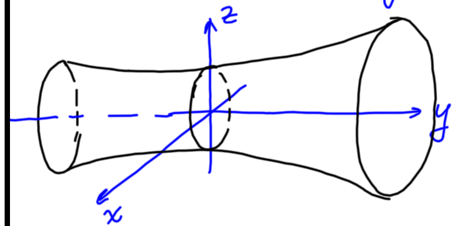
$$x^2 - y^2 + z^2 = 1$$

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

$x=k$: $\frac{-y^2+z^2}{1-k^2} = \frac{1-k^2}{1-k^2} \Rightarrow -\frac{y^2}{1-k^2} + \frac{z^2}{1-k^2} = 1$ hyperbola

$y=k$: $x^2+z^2 = 1+k^2 \Rightarrow \frac{x^2}{1+k^2} + \frac{z^2}{1+k^2} = 1$ ellipse

$z=k$: $x^2-y^2 = 1-k^2 \Rightarrow \frac{x^2}{1-k^2} - \frac{y^2}{1-k^2} = 1$ hyperbola

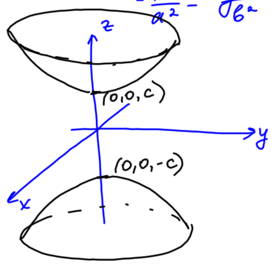


- **Hyperboloid of two sheets.** The quadric surface with equations

1. $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

are called hyperbolids of two sheets since two of its traces are hyperbolas and one is an ellipse, and its surface is two sheets.

$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, not defined when $|z|<c$



Traces:

$x=k$: $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{a^2}$ hyperbola

$y=k$: $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1 + \frac{k^2}{b^2}$ hyperbola

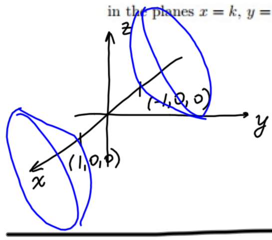
$z=k$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1$ ellipse

Example 3. Find the traces of the surface

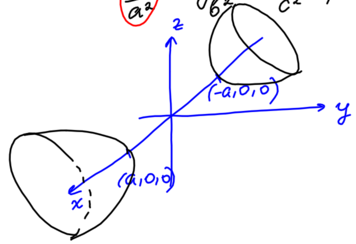
$$\frac{9x^2 - y^2 - z^2}{9} = \frac{9}{9}$$ – hyperboloid of two sheets.

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

$\frac{x^2}{1} - \frac{y^2}{9} - \frac{z^2}{9} = 1$, not defined for $|x|<1$



$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ not defined for $|x|<a$



Traces: $x=k$:

$$\frac{\frac{y^2}{b^2} + \frac{z^2}{c^2}}{\frac{k^2}{a^2}-1} = \frac{\frac{k^2}{a^2}-1}{\frac{k^2}{a^2}-1}$$

$\frac{y^2}{b^2(\frac{k^2}{a^2}-1)} + \frac{z^2}{c^2(\frac{k^2}{a^2}-1)} = 1$ ellipse

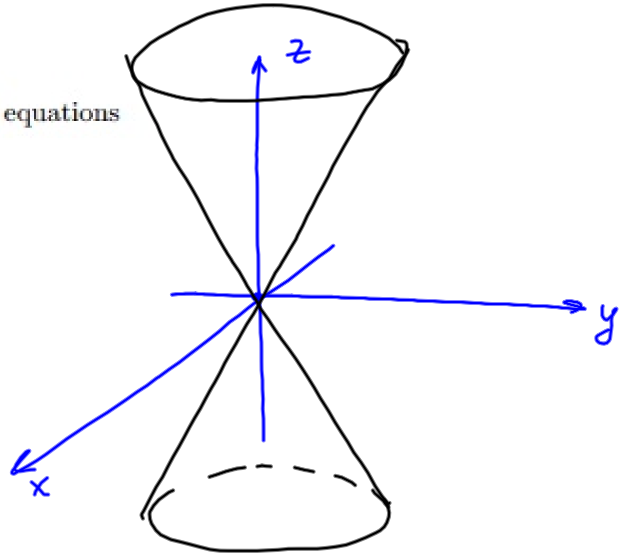
$y=k$: $\frac{x^2}{a^2(1+\frac{k^2}{b^2})} - \frac{z^2}{c^2(1+\frac{k^2}{b^2})} = 1$ hyperbola

$z=k$: $\frac{x^2}{a^2(1+\frac{k^2}{c^2})} - \frac{y^2}{b^2(1+\frac{k^2}{c^2})} = 1$ hyperbola

Cone. The quadric surface with equations

1. $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
2. $\frac{y^2}{b^2} = \frac{x^2}{a^2} + \frac{z^2}{c^2}$
3. $\frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$

are called cones.



If P is any point on the cone, then the line OP lies entirely on the cone. The traces in horizontal planes $z = k$ are ellipses and traces in vertical planes $x = k$ or $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Traces

$x=k$ $\quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = \frac{k^2}{a^2}$ hyperbola

$x=0$ $\quad \frac{z^2}{c^2} = \frac{y^2}{b^2}$ pair of lines $\frac{z}{c} = \frac{y}{b}$, $\frac{z}{c} = -\frac{y}{b}$

$y=k$ $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} = \frac{k^2}{b^2}$ hyperbola

$y=0$ $\quad \frac{z^2}{c^2} = \frac{x^2}{a^2}$ pair of lines $\frac{z}{c} = \frac{x}{a}$, $\frac{z}{c} = -\frac{x}{a}$

$z=k$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$ ellipse

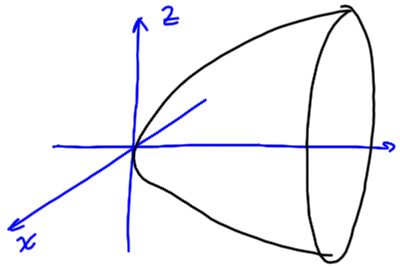
Paraboloids.

- **Elliptic paraboloids.** The quadric surface with equations

1. $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
2. $\frac{y}{b} = \frac{x^2}{a^2} + \frac{z^2}{c^2}$
3. $\frac{x}{a} = \frac{y^2}{b^2} + \frac{z^2}{c^2}$

are called **elliptic paraboloids** because its traces in horizontal planes $z = k$ are ellipses, whereas its in vertical planes $x = k$ or $y = k$ are parabolas.

$$\frac{y}{b} = \frac{x^2}{a^2} + \frac{z^2}{c^2}$$



$b > 0$

Traces:

$x = k$ $\quad \frac{y}{b} = \frac{z^2}{c^2} + \frac{k^2}{a^2}$ parabola

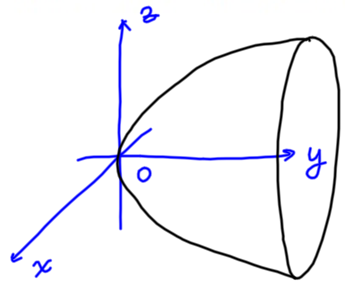
$y = k$ $\quad \frac{k}{b} = \frac{z^2}{c^2} + \frac{x^2}{a^2}$ ellipse

$z = k$ $\quad \frac{y}{b} = \frac{x^2}{a^2} + \frac{k^2}{c^2}$ parabola

Example 4. Find the traces of the surface

$$y = x^2 + z^2 \quad \text{– elliptic paraboloid}$$

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.



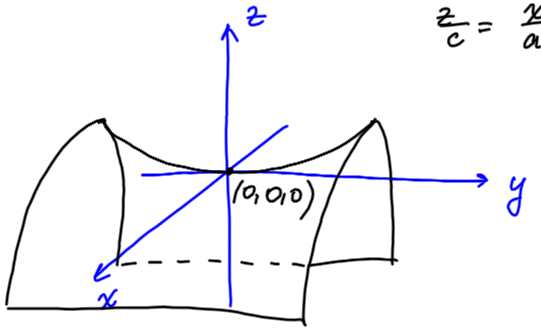
$\underline{y = k}$ $\quad x^2 + z^2 = k$ circle

- **Hyperbolic paraboloids.** The quadric surface with equations

1. $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$
2. $\dfrac{y}{b} = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$
3. $\dfrac{x}{a} = \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$

are called **hyperbolic paraboloids** because its traces in horizontal planes $z = k$ are hyperbolas, whereas its in vertical planes $x = k$ or $y = k$ are parabolas.

$\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$, $c > 0$

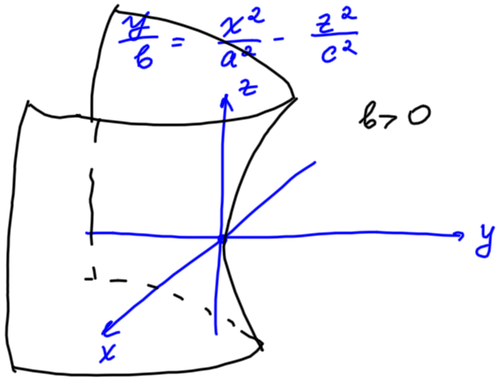


Traces:

$\underline{x=k}$: $\dfrac{z}{c} = \dfrac{k^2}{a^2} - \dfrac{y^2}{b^2}$ parabola

$\underline{y=k}$: $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{k^2}{b^2}$ parabola

$\underline{z=k}$: $\dfrac{k}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ hyperbola



traces:

$\underline{x=k}$ $\dfrac{y}{b} = \dfrac{k^2}{a^2} - \dfrac{z^2}{c^2}$ parabola

$\underline{y=k}$ $\dfrac{k}{b} = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$ hyperbola

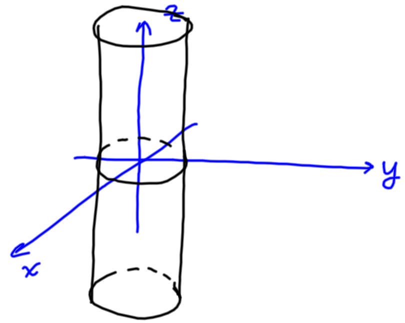
$\underline{z=k}$ $\dfrac{y}{b} = \dfrac{x^2}{a^2} - \dfrac{k^2}{c^2}$ parabola

Quadric cylinders. When one of the veriables is missing from the equation of a surface, then a surface is a cylinder.

- The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents the elliptic cylinder

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

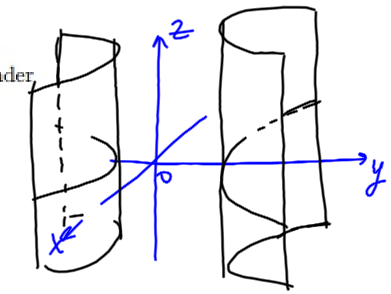
$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$



- The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents the hyperbolic cylinder

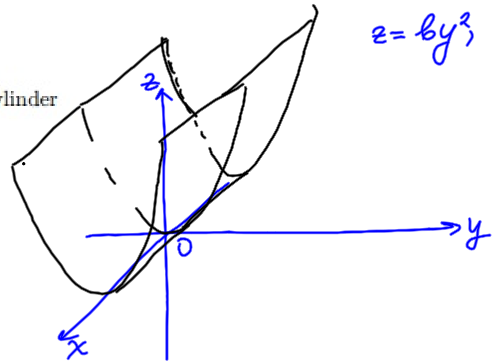
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$



- The equation $y = ax^2$ represents the parabolic cylinder

$x = by^2$
$y = cz^2$
$x = cz^2$
$\cancel{z} = ax^2$
$z = by^2$



$z = by^2;\ b > 0$

Example 5. Classify the surface

$$4x^2 - y^2 + z^2 + 8x + 8z + 24 = 0$$

and sketch it.

$$(4x^2+8x) - y^2 + (z^2+8z) + 24 = 0$$

Complete squares

$$4(x^2+2x+1) - y^2 + (z^2+\overset{4\cdot 2}{8z}+4^2) - 4\cdot 1 - 4^2 + 24 = 0$$

$$\frac{4(x+1)^2 - y^2 + (z+4)^2}{-4} = \frac{-4}{-4}$$

$$\boxed{-(x+1)^2 + \frac{y^2}{4} - \frac{(z+4)^2}{4} = 1}$$

hyperboloid of two sheets

$(-1, 0, -4)$ "new origin"

is not defined when $|y| < \sqrt{4}$

$|y| < 2$