Section 13.1 **Vector functions and space curves.**

Let $\mathbf{r}$ be a **vector function** whose range is a set of three-dimensional vectors.

$$\mathbf{r}(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Functions f, g, and h are real-valued functions called the **component functions** of $\mathbf{r}$.

The domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined.

Example 1. Find the domain of the vector function $\mathbf{r}(t) = \left\langle \sqrt{9-t}, \sqrt{t-2}, \frac{e^t}{t-5} \right\rangle$.

Definition. If $\mathbf{r}(t) =< f(t), g(t), h(t) >$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

provided the limits of the component function exist.

Example 2. Find the limit

$$\lim_{t\to 0} \left\langle \frac{1-\cos t}{t}, t^3, e^{-1/t^2} \right\rangle.$$

Definition. A vector function $\mathbf{r}$ is **continuous at** a if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$.

$\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t), \quad y = g(t) \quad z = h(t)$$

ant t varies throughtout the interval I, is called a **space curve**. Equations

$$x = f(t), \quad y = g(t) \quad z = h(t)$$

are called **parametric equations of C** and t is called a **parameter**.

Example 3. Sketch the curve with the given vector equation.

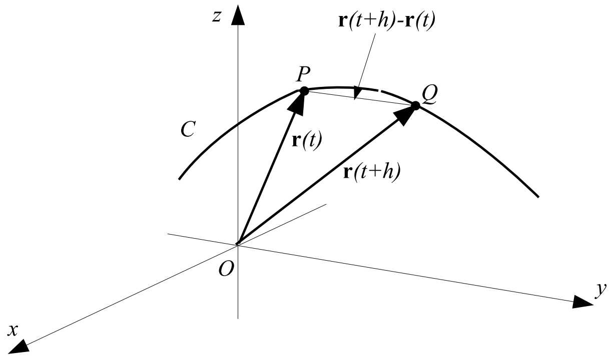
1. $\mathbf{r}(t) =< 1 - t, t, t - 2 >$

2. $\mathbf{r}(t) =< \cos 4t, t, \sin 4t >$

3. $\mathbf{r}(t) =< \cos t, \sin t, \sin 5t >$

Derivatives and integrals. The **derivative r′** of a vector function **r** is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \vec{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector**

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Theorem. If $\mathbf{r}(t) =< f(t), g(t), h(t) >$, then

$$\mathbf{r}'(t) =< f'(t), g'(t), h'(t) >$$

Example 4. Find the derivative of the vector function $\mathbf{r}(t) = \ln(4 - t^2)\mathbf{i} + \sqrt{1+t}\mathbf{j} - 4e^{3t}\mathbf{k}$.

Example 5. At what point do the curves $\mathbf{r}_1(t) =< t, 1-t, 3+t^2 >$ and $\mathbf{r}_2(s) =< 3-s, s-2, s^2 >$ intersect? Find their angle of intersection.

Theorem. Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(t)$

If $|\mathbf{r}(t)| = c$, where c is a constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

The **definite integral** of a continuous vector function $\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle$

The Fundamental Theorem of Calculus for continuous vector functions says that

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$. We use the notation $\int \mathbf{r}(t)dt$ for indefinite integrals (antiderivatives).

Example 6. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) =< \sin t, -\cos t, 2t >$ and $\mathbf{r}(0) = \mathbf{i}+\mathbf{j}+2\mathbf{k}$.