Section 13.1 **Vector functions and space curves.**

Let $\mathbf{r}$ be a vector function whose range is a set of three-dimensional vectors.

$$\boxed{\mathbf{r}(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}}$$

Functions f, g, and h are real-valued functions called the **component functions of r**.

The domain of $\mathbf{r}$ consists of all values of t for which the expression for $\mathbf{r}(t)$ is defined.

$\sqrt[2n]{f(t)}$, $f(t) \geq 0$

$\ln[f(t)]$, $f(t) > 0$

$\frac{P(t)}{Q(t)}$, $Q(t) \neq 0$

$\tan t$, $t \neq \frac{\pi}{2} + \pi n$

$\cot t$, $t = \pi n$,

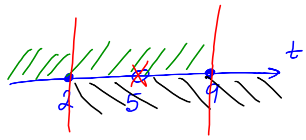
$n = 0, \pm 1, \pm 2, \ldots$

Example 1. Find the domain of the vector function $\mathbf{r}(t) = \left\langle \sqrt{9-t}, \sqrt{t-2}, \frac{e^t}{t-5} \right\rangle$.

$\sqrt{9-t} \rightarrow 9 - t \geq 0$ or $t \leq 9$

$\sqrt{t-2} \rightarrow t - 2 \geq 0$ or $t \geq 2$

$\frac{e^t}{t-5} \rightarrow t \neq 5$



$$\boxed{[2,5) \cup (5,9]}$$

Definition. If $\mathbf{r}(t) =< f(t), g(t), h(t) >$, then

$$\boxed{\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle}$$

provided the limits of the component function exist.

Example 2. Find the limit

$$\lim_{t\to 0} \left\langle \frac{1-\cos t}{t}, t^3, e^{-1/t^2} \right\rangle = < \lim_{t\to 0} \frac{1-\cos t}{t}, \lim_{t\to 0} \cancel{t^3}^{\,0}, \underbrace{\lim_{t\to 0} e^{-\frac{1}{t^2}}}_{\lim_{s\to -\infty} e^{s} = 0} >$$

L.H. Rule $\left[\frac{0}{0}\right]$

$\lim_{t\to 0} \frac{-1}{t^2} = -\infty$

$$= < \lim_{t\to 0} \frac{\cancel{\sin t}^{\,0}}{1}, 0, 0 > = \boxed{<0,0,0>}$$

Definition. A vector function $\mathbf{r}$ is **continuous at** a if $\boxed{\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)}$.

$\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

Space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t), \quad y = g(t) \quad z = h(t)$$

ant t varies throughtout the interval I, is called a **space curve**. Equations

$$x = f(t), \quad y = g(t) \quad z = h(t)$$

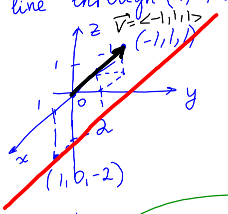
are called **parametric equations of C** and t is called a **parameter**.

Example 3. Sketch the curve with the given vector equation.

1. $\mathbf{r}(t) = < 1-t, t, t-2 >$

$$\begin{cases} x = 1-t \\ y = t \\ z = t-2 \end{cases}$$

line through $(1,0,-2)$ parallel to $<-1,1,1>$



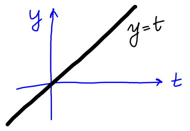
2. $\mathbf{r}(t) = < \cos 4t, t, \sin 4t >$

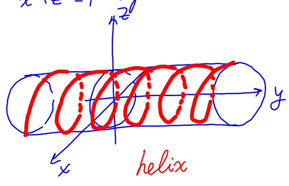
$$\begin{cases} x = \cos 4t \\ y = t \\ z = \sin 4t \end{cases}$$

$\sin^2 x + \cos^2 x = 1$

$\underbrace{\sin^2 4t}_{z^2} + \underbrace{\cos^2 4t}_{x^2} = 1$

$x^2 + z^2 = 1$ cylinder



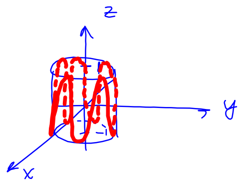


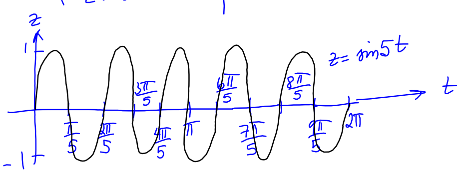
3. $\mathbf{r}(t) = < \cos t, \sin t, \sin 5t >$

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = \sin 5t \end{cases}$$

$x^2 + y^2 = 1$ - cylinder

$-1 \le z \le 1$





Derivatives and integrals. The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is

$$\boxed{\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}}$$

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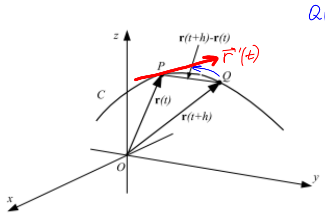
$P(x(t), y(t), z(t))$

$Q(x(t+h), y(t+h), z(t+h))$



The vector $\mathbf{r}'(t)$ is called the tangent vector to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \vec{0}$. The tangent line to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}$$

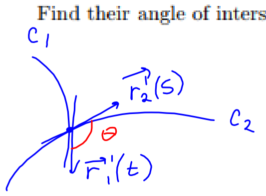
Theorem. If $\mathbf{r}(t) = < f(t), g(t), h(t) >$, then

$$\boxed{\mathbf{r}'(t) = < f'(t), g'(t), h'(t) >}$$

Example 4. Find the derivative of the vector function $\mathbf{r}(t) = \ln(4-t^2)\mathbf{i} + \sqrt{1+t}\mathbf{j} - 4e^{3t}\mathbf{k}$.

$$\boxed{\vec{r}\,'(t) = < \frac{-2t}{4-t^2}, \frac{1}{2}(1+t)^{-1/2}, -12e^{3t} >}$$

Example 5. At what point do the curves $\mathbf{r}_1(t) = < t, 1-t, 3+t^2 >$ and $\mathbf{r}_2(s) = < 3-s, s-2, s^2 >$ intersect? Find their angle of intersection.



θ is an angle of intersection

Point of intersection:

$$\begin{cases} t = 3-s \\ 1-t = s-2 \\ 3+t^2 = s^2 \end{cases} \quad s+t=3$$

$$\begin{cases} t = 3-s \\ 3+t^2 = s^2 \end{cases}$$

$$3+(3-s)^2 = s^2$$

$$3+9-6s+\cancel{s^2} = \cancel{s^2}$$

$$\boxed{s=2}\ \boxed{t=1}$$

$$\vec{r}_1(1) = <1,0,4> = \vec{r}_2(2)$$

$$\boxed{(1,0,4)}$$

Tangent vectors:

$\vec{r}_1'(t) = <1,-1,2t>$ | $\vec{r}_1'(1) = <1,-1,2>$ tangent vector to C_1 @ $(1,0,4)$

$\vec{r}_2'(s) = <-1,1,2s>$ | $\vec{r}_2'(2) = <-1,1,4>$ tangent vector to C_2 @ $(1,0,4)$

Angle:

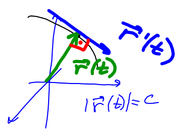
$$\cos\theta = \frac{\vec{r}_1'(1)\cdot\vec{r}_2'(2)}{|\vec{r}_1'(1)|\cdot|\vec{r}_2'(2)|} = \frac{<1,-1,2>\cdot<-1,1,4>}{\sqrt{1+1+4}\,\sqrt{1+1+16}} = \frac{6}{\sqrt{6}\sqrt{18}} = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)}$$

Theorem. Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\boxed{\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)}$
5. $\boxed{\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)}$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

If $|\mathbf{r}(t)| = c$, where c is a constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.



The **definite integral** of a continuous vector function $\boxed{\int_a^b \mathbf{r}(t)dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle}$

The Fundamental Theorem of Calculus for continuous vector functions says that

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$. We use the notation $\int \mathbf{r}(t)dt$ for indefinite integrals (antiderivatives).

Example 6. Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = < \sin t, -\cos t, 2t >$ and $\mathbf{r}(0) = \mathbf{i}+\mathbf{j}+2\mathbf{k}$. $= <1,1,2>$

$$\vec{r}(t) = \int \vec{r}'(t)dt = <\int \sin t\,dt, -\int \cos t\,dt, \int 2t\,dt>$$

$$= <-\cos t + C_1, -\sin t + C_2, t^2 + C_3>$$

$$\vec{r}(0) = <-1+C_1, C_2, C_3> = <1,1,2>$$

$$-1+C_1 = 1 \Rightarrow C_1 = 2$$

$$C_2 = 1$$

$$C_3 = 2$$

$$\boxed{\vec{r}(t) = <-\cos t + 2, -\sin t + 1, t^2+2>}$$