Section 13.3 **Arc length and curvature.**

The length of a space curve with the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$, if the curve is traversed exactly once as t increases from a to b, is

$$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_a^b |\mathbf{r}'(t)|\,dt}$$

Example 1. Find the length of the curve given by the vector function $\mathbf{r}(t) =< e^t, e^t \sin t, e^t \cos t >, 0 \le t \le 2\pi$.

$x(t) = e^t$	$x'(t) = e^t$
$y(t) = e^t \sin t$	$y'(t) = e^t \sin t + e^t \cos t = e^t(\sin t + \cos t)$
$z(t) = e^t \cos t$	$z'(t) = e^t(\cos t - \sin t)$

$$L = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{e^{2t} + e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{e^{2t}\left[1 + \sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + \cos^2 t - \cancel{2\sin t \cos t} + \sin^2 t\right]}\,dt$$

$$\cos^2 t + \sin^2 t = 1$$

$$= \int_0^{2\pi} \sqrt{3e^{2t}}\,dt = \int_0^{2\pi} e^t\sqrt{3}\,dt = \sqrt{3}\,e^t\Big|_0^{2\pi} = \boxed{\sqrt{3}\left(e^{2\pi} - 1\right)}$$

A curve given by a vector function $\mathbf{r}(t)$ on an interval I is called smooth if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \vec{0}$ (except possibly at any endpoints of I). A curve that is made up of a finite number of smooth pieces is called piecewise smooth. The arc length formula holds for piecewise-smooth functions.

A single curve C can be represented by more than one vector function. Arc length is independent of the parametrization that is used.

Suppose that C is a piecewise-smooth curve given by a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \le t \le b$ and at least one of f, g, h is one-to-one on (a, b). We define its arc length function s by

$$\boxed{s(t) = \int_a^t |\mathbf{r}'(u)|\,du} = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

It is often useful to **parametrize a curve with respect to arc length.**

1

Example 2. Reparametrize the curve $\mathbf{r}(t) = \langle 1 + 2t, 3 + t, -5t\rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$\vec{r}\,'(t) = <2, 1, -5>, \quad |\vec{r}\,'(t)| = \sqrt{4 + 1 + 25} = \sqrt{30}$$

Ark length function:

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{30}\,du = \sqrt{30}\,u\Big|_0^t = \sqrt{30}\,t$$

$$s = \sqrt{30}\,t \Rightarrow t = \frac{s}{\sqrt{30}}$$

$$\boxed{\vec{r}(s) = <1 + \frac{2s}{\sqrt{30}}, 3 + \frac{s}{\sqrt{30}}, \frac{-5s}{\sqrt{30}}>}, \quad |\vec{r}\,'(s)| = 1$$

Curvature.
If C is a smooth curve defined by the vector function $\mathbf{r}$, then $\mathbf{r}'(t) \neq \vec{0}$. The unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

indicates the direction of the curve.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. We define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

Definition. The **curvature** of a curve is

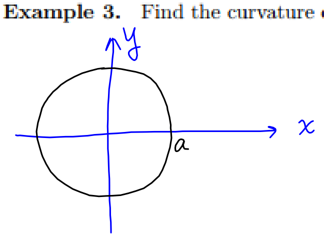
$$\kappa = \left|\frac{d\mathbf{T}(t)}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

Since $\frac{ds}{dt} = |\mathbf{r}'(t)|$, then

$$\boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$

Example 3. Find the curvature of a circle of radius a.



$x^2+y^2=a^2$

$x = a\cos t$

$y = a\sin t$, $0 \le t \le 2\pi$

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$

$\vec{r}\,'(t) = \langle -a\sin t, a\cos t\rangle$, $|\vec{r}\,'(t)| = a$

unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t\rangle$

$\vec{T}\,'(t) = \langle -\cos t, -\sin t\rangle$, $|\vec{T}\,'(t)| = 1$

$$\boxed{\kappa(t) = \frac{1}{a}}$$

Theorem. The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\boxed{\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}}$$

Example 4. Find the curvature of the curve given by the vector function

$$\mathbf{r}(t) = (t^2+2)\mathbf{i} + (t^2-4t)\mathbf{j} + 2t\mathbf{k}.$$

$\vec{r}\,'(t) = \langle 2t, 2t-4, 2\rangle$, $|\vec{r}\,'(t)| = \sqrt{4t^2+(2t-4)^2+4}$

$= \sqrt{4t^2+4t^2-16t+16+4} = \sqrt{8t^2-16t+20}$

$4(2t^2-4t+5)$

$|\vec{r}\,'(t)| = 2\sqrt{2t^2-4t+5}$

$\vec{r}\,''(t) = \langle 2, 2, 0\rangle$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2t-4 & 2 \\ 2 & 2 & 0 \end{vmatrix} = -4\vec{i} + 4\vec{j} + \vec{k}\,[4t - 2(2t-4)]$$

$= \langle -4, 4, 8\rangle$

$|\vec{r}\,'\times\vec{r}\,''| = \sqrt{16+16+64} = 4\sqrt{6}$

$$\kappa(t) = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{4\sqrt{6}}{8(2t^2-4t+5)^{3/2}} = \boxed{\frac{\sqrt{6}}{2(2t^2-4t+5)^{3/2}}}$$

For the special case of a plane curve with equation $y = f(x)$, the curvature

$$\boxed{\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}}$$

Example 5. Find the curvature of the function $y = \sqrt{x}. = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}}$$

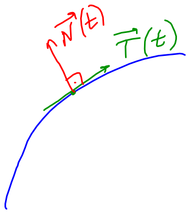
$$\kappa(x) = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}} = \frac{\frac{1}{4x^{3/2}}}{\left(1+\left[\frac{1}{2\sqrt{x}}\right]^2\right)^{3/2}}$$

$$= \frac{\frac{1}{4x^{3/2}}}{\left(1+\frac{1}{4x}\right)^{3/2}} = \frac{\frac{1}{4x^{3/2}}}{\left(\frac{4x+1}{4x}\right)^{3/2}} = \frac{(4\cancel{x})^{3/2}}{4\cancel{x^{3/2}}(4x+1)^{3/2}}$$

$$= \boxed{\frac{2}{(4x+1)^{3/2}}}$$

Normal Vectors.
We define the unit normal vector as

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.}$$



Vectors $\mathbf{N}(t)$ and $\mathbf{T}(t)$ are orthogonal.

Example 6. Find the unit normal vector for the curve given by $\mathbf{r}(t) = \left\langle t, \frac{t^2}{2}, t^2 \right\rangle$.

$$\vec{r}\,'(t) = \langle 1, t, 2t \rangle, \quad |\vec{r}\,'(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$$

unit tangent vector $\vec{T}(t) = \left\langle \frac{1}{\sqrt{1+5t^2}}, \frac{t}{\sqrt{1+5t^2}}, \frac{2t}{\sqrt{1+5t^2}} \right\rangle$

$$\vec{T}\,'(t) = \left\langle -\frac{1}{2}(1+5t^2)^{-3/2}(10t), \frac{\sqrt{1+5t^2} - t\,\frac{1}{2}(1+5t^2)^{-1/2}(10t)}{1+5t^2}, \frac{\sqrt{1+5t^2} - 5t^2(1+5t^2)^{-1/2}}{1+5t^2}\cdot 2 \right\rangle$$

$$= \left\langle -\frac{5t}{(1+5t^2)^{3/2}}, \frac{1}{(1+5t^2)^{3/2}}, \frac{2}{(1+5t^2)^{3/2}} \right\rangle$$

$$\sqrt{1+5t^2} - \frac{5t^2}{\sqrt{1+5t^2}} = \frac{1+5t^2-5t^2}{\sqrt{1+5t^2}} = \frac{1}{\sqrt{1+5t^2}}$$

$$|\vec{T}\,'(t)| = \sqrt{\frac{25t^2}{(1+5t^2)^3} + \frac{1}{(1+5t^2)^3} + \frac{4}{(1+5t^2)^3}} = \frac{\sqrt{25t^2+5}}{(1+5t^2)^{3/2}}$$

unit normal vector $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$

$$= \boxed{\left\langle -\frac{5t}{\sqrt{25t^2+5}}, \frac{1}{\sqrt{25t^2+5}}, \frac{2}{\sqrt{25t^2+5}} \right\rangle}$$