

Chapter 14. Partial derivatives.  
Section 14.1 Functions of several variables.

**Definition.** Let  $D \subset \mathbb{R}^2$ . A **function  $f$  of two variables** is a rule that assigns to each ordered pair  $(x, y)$  in  $D$  a unique real number denoted by  $f(x, y)$ . The set  $D$  is the **domain** of  $f$  and its **range** is the set of values that  $f$  takes on, that is,  $\{f(x, y) | (x, y) \in D\}$ .

We write  $z = f(x, y)$  to make explicit the value taken on by  $f$  at the general point  $(x, y)$ . The variables  $x$  and  $y$  are **independent variables** and  $z$  is **dependent variable**.

If a function  $f$  is given by a formula and no domain is specified, then the domain of  $f$  is understood to be the set of all pairs  $(x, y)$  for which the given expression is well-defined real number.

**Example 1.** Find the domain and the range of the function  $f(x, y) = x^2 \ln(x - y)$  and evaluate  $f(e, 0)$ .

**Definition.** If  $f$  is a function of two variables with domain  $D$ , the **graph** of  $f$  is the set

$$S = \{(x, y, z) \in \mathbb{R}^3 | z = f(x, y), (x, y) \in D\}.$$

**Example 2.** Sketch the graph of the function  $f(x, y) = 3 - x^2 - y^2$ .

**Definition.** The **level curves** of a function  $f$  of two variables are the curves with equations  $f(x, y) = k$ , where  $k$  is a constant (in the range of  $f$ ).

A level curve  $f(x, y) = k$  is the locus of all points at which  $f$  takes on a given value  $k$ . In other words, it shows where the graph of  $f$  has height  $k$ .

**Example 3.** Describe the level curves for the following functions.

1.  $f(x, y) = -x + 4y$

2.  $f(x, y) = x^2 - y^2$

**Functions of three or more variables.**

A **function of three variables**,  $f$ , is a rule that assigns to each ordered triple  $(x, y, z)$  in a domain  $D \subset \mathbb{R}^3$  a unique real number denoted by  $f(x, y, z)$ .

We can get some information about  $f$  by examining its **level surfaces**, which are surfaces with equations  $f(x, y, z) = k$ , where  $k$  is a constant. If the point  $(x, y, z)$  moves along a level surface, the value of  $f(x, y, z)$  remains fixed.

**Example 4.** Find the domain of the function  $f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$ .

**Example 5.** Describe the level surfaces of the function  $f(x, y, z) = x^2 - y^2 + z^2$ .

A **function of  $n$  variables** is a rule that assigns a number  $z = f(x_1, x_2, \dots, x_n)$  to an  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  of real numbers. The notation

$$f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

is used to signify that  $f$  is a real valued function whose domain  $D$  is a subset of  $\mathbb{R}^n$ .